MST121 Chapter A0

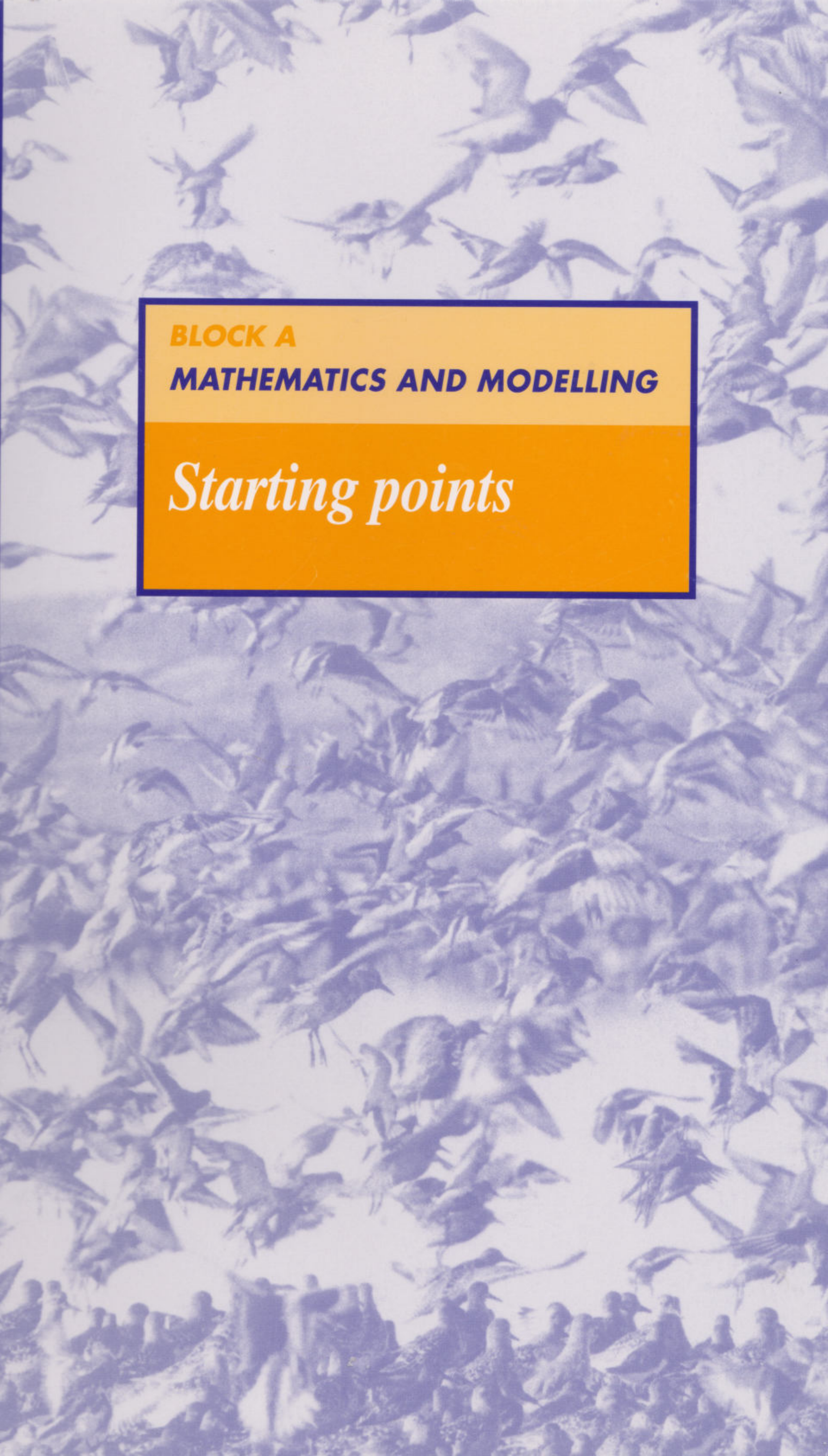
The Open University

A first level Interdisciplinary course

Using Mathematics

BLOCK A
MATHEMATICS AND MODELLING

Starting points



CHAPTER

A0

MST121 Chapter A0

The Open University

A first level Interdisciplinary course

Using Mathematics

CHAPTER A0

BLOCK A

MATHEMATICS AND MODELLING

Starting points

Prepared by the course team

About this course

This course, MST121 *Using Mathematics*, and the courses MU120 *Open Mathematics* and MS221 *Exploring Mathematics* provide a flexible means of entry to university-level mathematics. Further details may be obtained from the address below.

MST121 uses the software program Mathcad (MathSoft, Inc.) and other software to investigate mathematical and statistical concepts and as a tool in problem solving. This software is provided as part of the course.

This publication forms part of an Open University course. Details of this and other Open University courses can be obtained from the Course Reservations Centre, PO Box 724, The Open University, Milton Keynes, MK7 6ZS, United Kingdom: tel. +44 (0)1908 653231, e-mail ces-gen@open.ac.uk

Alternatively, you may visit the Open University website at http://www.open.ac.uk where you can learn more about the wide range of courses and packs offered at all levels by the Open University.

To purchase this publication or other components of Open University courses, contact Open University Worldwide Ltd, The Berrill Building, Walton Hall, Milton Keynes, MK7 6AA, United Kingdom: tel. +44 (0)1908 858785, fax +44 (0)1908 858787, e-mail ouwenq@open.ac.uk or website http://www.ouw.co.uk

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 1997. New edition 2001. Reprinted 2001.

Edited, designed and typeset by The Open University, using the Open University TEX System.

Printed and bound in the United Kingdom by The Burlington Press, Foxton, Cambridge CB2 6SW.

ISBN 0 7492 9314 4

2.2

Contents

Introduction to Block A

The title of this block, *Mathematics and modelling*, indicates one of the principal themes of the course, namely 'mathematical modelling'. This phrase describes the use of mathematics to represent, or 'model', objects and processes in the real world and, thereby, to try to obtain new information about these objects and processes.

Effective mathematical modelling requires several skills. First, you must acquire appropriate mathematical skills and develop facility at using them. Next, you need practice and experience at expressing real-world problems as mathematical problems, and an understanding of which mathematical techniques might be applied to solve these problems. Finally, you need appropriate computing skills and an appreciation of when the computer should be used in the solution of problems.

Chapter A0 provides a foundation for the development of your mathematical skills, and introduces the mathematical software that you will use in the course. It takes you through the installation of the software package Mathcad and associated course files, and provides revision material designed to indicate various ways in which Mathcad can support your mathematical work.

'Sequence' is the mathematical name for an ordered list.

Chapter A1 deals with certain types of sequences that can be used to represent real-life processes as diverse as mortgage repayments and animal populations. You will learn how to recognise such sequences and find formulas which make it possible to describe how the sequences behave.

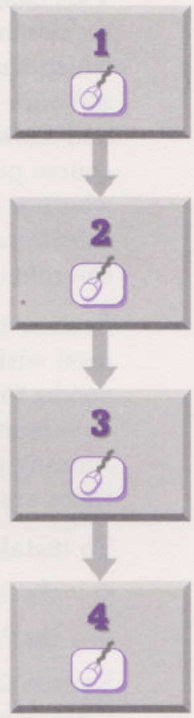
Chapter A2 presents the use of algebraic techniques for representing lines and circles. These geometric objects are fundamental to the description of certain aspects of the real world, such as the construction of maps. Therefore, being able to perform calculations with lines and circles is important to our understanding of the real world.

Chapter A3 introduces the mathematical object known as a *function*, which can be described informally as 'a process for converting inputs to outputs'. Many real-world problems can be expressed conveniently in terms of appropriate functions, so much mathematics is devoted to developing techniques for dealing with functions. In particular, this chapter shows how functions and their graphs can be used to solve problems involving areas and volumes.

Each of the chapters in this block includes teaching material which shows how Mathcad can be used with the topic being studied. This material is designed to help you understand when use of the computer is appropriate and to increase your range of computer skills in a systematic way.

Study guide

There are four sections to this chapter, which are intended to be studied consecutively, and an appendix that you should read before moving on to Chapter A1. Each section should take between two and three hours of study. All sections require the use of this main text and the computer, Section 1 requires the MST121/MS221 CD-ROM, and Sections 3 and 4 require the use of the calculator. For Sections 3 and 4 you may find it helpful to have the *Revision Pack* to hand.



The pattern of study for each session might be as follows.

Study session 1: Section 1 and Subsection 2.1.

Study session 2: The remainder of Section 2.

Study session 3: Section 3.

Study session 4: Section 4.

Solutions to activities that involve arithmetic or algebraic calculation start on page 43. Such activities based on the computer have on-screen solutions.

In this chapter you will need to use the following Mathcad files.

121A0-01 Documents, expressions and text

121A0-02 Rearranging documents

121A0-03 Calculating with Mathcad

121A0-04 Using variables

121A0-05 Rearranging algebraic expressions

Instructions for using these files are contained in Activities in this text.

All the Mathcad skills that you practise in this chapter will be revisited in later chapters.

Introduction

This chapter has several important aims. It summarises the mathematical techniques in arithmetic and algebra on which the course builds, much of which should be familiar to you. It also introduces the mathematical software package Mathcad, which is an important resource for your study, and shows you how to use Mathcad to do arithmetic and algebra. As the course progresses, there will be many occasions where you can choose whether to carry out a particular piece of mathematics by hand, by calculator or by using Mathcad, so you need to begin to think here about the role of Mathcad in your work.

Section 1 explains how to install Mathcad, and the course files which are used with Mathcad, onto your computer. It is assumed here that you have access to a computer that satisfies the course specification, and that you have some experience of the *Windows* (*95*, *98*, etc.) environment and of running applications. *Windows* is used in a very straightforward way, and there are explicit instructions on what you have to do.

To install the software, you need this chapter and:

◇ the MST121/MS221 CD-ROM;

◇ the Mathcad installation password, supplied with the course materials.

Mathcad is licensed, and requires you to use the password to install it.

Section 2 guides you through some first steps in using Mathcad. The two main uses of Mathcad in the course will be to perform routine calculations and to carry out investigations that would not otherwise be possible.

Both Sections 3 and 4 build on material in the *Revision Pack*, and you may like to consult this at times while studying this chapter.

Section 3 looks at numbers and calculations. Numbers are classified into various categories, such as *integer* (whole number), *rational* (fraction) and *irrational* (cannot be expressed as a fraction). Then the basic rules of arithmetic, including powers, are reviewed in preparation for seeing how calculations are done in Mathcad.

Section 4 shifts attention to algebra. It briefly reviews the rules for rearranging algebraic expressions, and then the procedures for solving linear and quadratic equations. You will discover that Mathcad can perform algebraic operations, and you will see that it can deal with cases which would be difficult to do by hand.

1 Installing the course software

1.1 Installing Mathcad

Switch on your computer and start running Windows.

When the screen settles down, you should see something like this, known as a desktop.

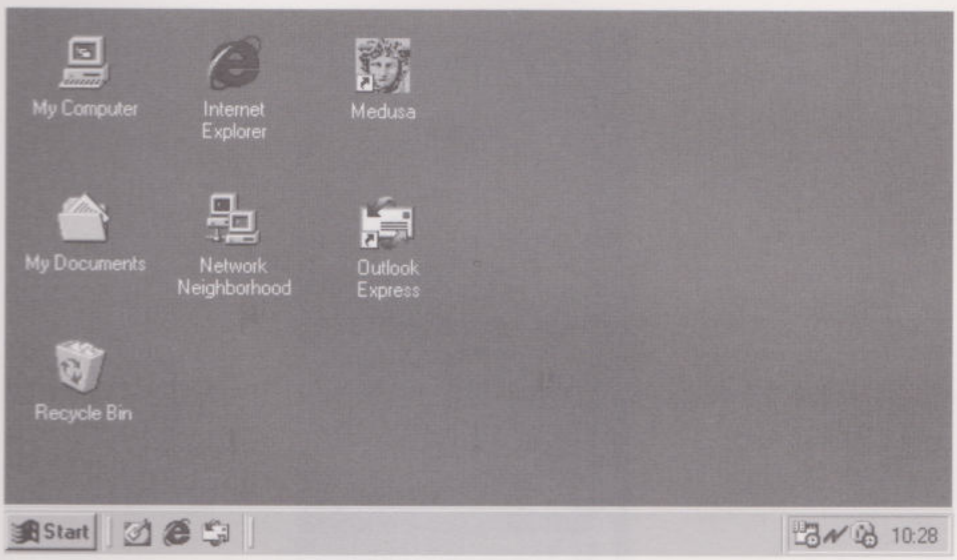


Figure 1.1 *Windows* desktop

It is assumed that your machine is operating under *Windows 95*, *98*, or a later version. Figures, such as Figure 1.1, that show *Windows* screens are taken from *Windows 98* (classic style). If you have a different version of *Windows*, then your screen will look slightly different, but it will operate in a similar way.

Windows is a very flexible operating system, and you might have many more features visible on the desktop than are shown in Figure 1.1. The only feature used here, however, is the **Start** menu, which appears when you click on the **Start** button on the **task bar** at the foot of the screen. (Exceptionally, the task bar may be hidden below the bottom of the screen. If so, you can drag it up to its usual position.)

Installing Mathcad is Stage 1 of a two-stage installation process. Installation means copying the MST121/MS221 version of Mathcad from the supplied CD-ROM onto your hard disk. Even if you already have a version of Mathcad on your computer, you should install the MST121/MS221 version, as described below. Versions of Mathcad may differ, and all course references are to the MST121/MS221 version.

To ensure successful Mathcad installation, your computer needs to satisfy the following three conditions:

- at least 15MB of free space on your hard disk;
- no *active* virus-checking software;
- no applications running.

If you have any worries about these conditions, and installation subsequently fails, then consult a course Stop Press for information about whom to contact.

The Mathcad installation password is supplied with the course materials.

To install Mathcad, you will need the CD-ROM labelled with the course codes MST121 and MS221, and the Mathcad installation password. You will also need to know which letter is associated with your CD-ROM drive; often it is the letter D.

Activity 1.1 Installing Mathcad

You might find it helpful to read through this activity before carrying out the various steps.

First make sure that your screen looks something like Figure 1.1. If there are any applications running, then close them now.

(a) Put the CD-ROM into the CD-ROM drive.

(b) Use the mouse to open the **Start** menu, and to select the **Run** option (Figure 1.2).

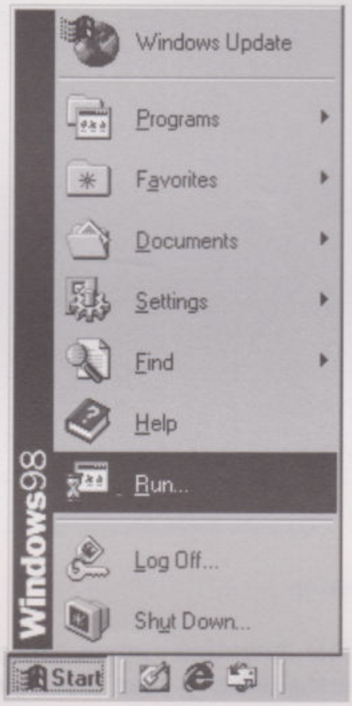


Figure 1.2 **Start** menu

You will see a dialog box (Figure 1.3), with the cursor flashing in the **Open** box.

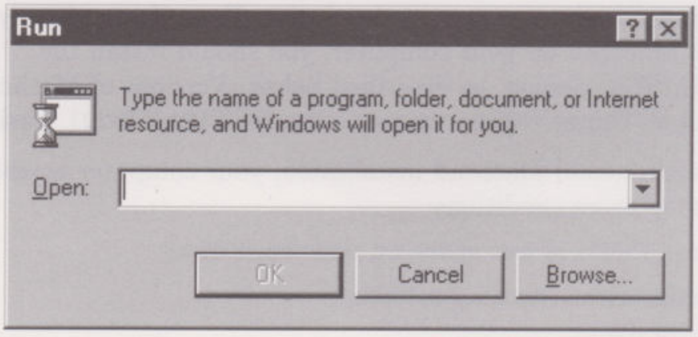


Figure 1.3 Run dialog box

(c) Type `d:\setup` in the **Open** box. (The CD-ROM drive is often drive D, so this is the letter used in these instructions, but if your drive has another letter, such as E, F or G, then use that; for example, type `e:\setup`.) Then click on the **OK** button.

The initial installation screen will appear (Figure 1.4).

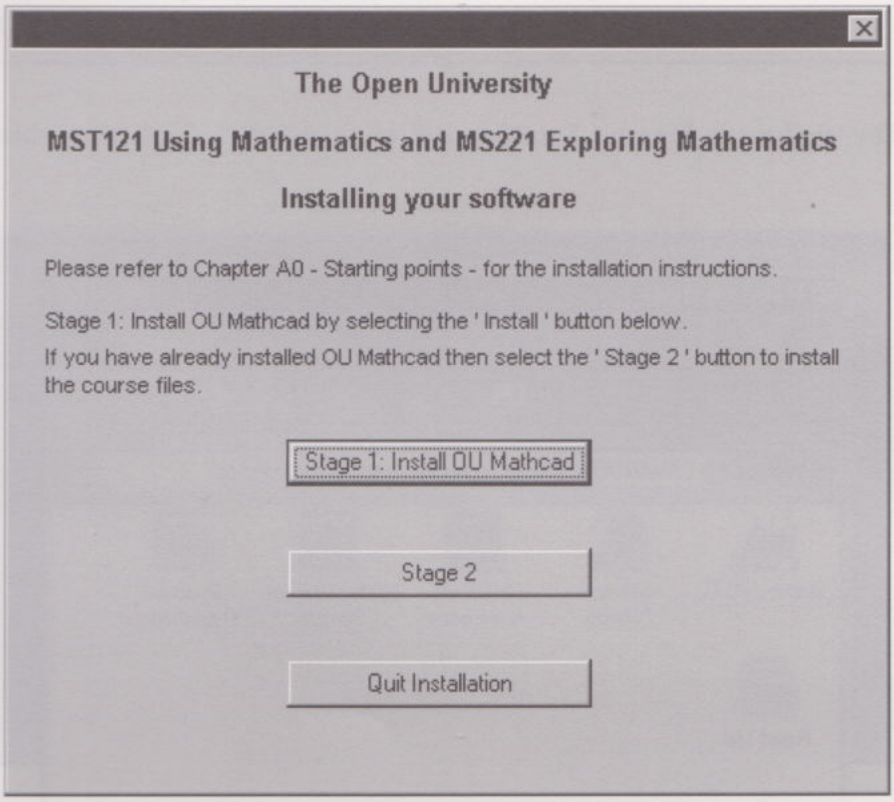


Figure 1.4 Initial installation screen

(d) Click on the **Stage 1: Install OU Mathcad** button.

(e) The next screen asks you to enter the installation password. Type it in, then click on the **OK** button.

The Mathcad installation password is supplied with the course materials.

(f) Read the 'Licence Warning' screen which follows, and click on the **Accept** button to proceed.

Installation will then begin. You should follow the instructions as you see them on the screen; however, **do not read** the **Mathcad 5.0 Installation Release Notes** unless you are an expert in such systems. These notes are very technical and probably do not concern your set-up.

(g) Simply click on the **OK** button in the Welcome dialog box to move on.

(h) You will be asked in which directory you want Mathcad to be installed. Conventionally, drive C is the hard disk and the place to install Mathcad. Unless you are an expert and deliberately want to arrange things differently, accept the default path **C:\WINMCAD** which you will find already typed in the Path box. To do this, just click on the **OK** button.

(i) From now on, everything is automatic, though the screen is a little busy. In the window labelled **Decompressing files** you may be able to make out a red bar extending to the right and underneath it a counter. These tell you how close installation is to completion.

(j) Eventually, a window will appear saying **Installation was Successful**. Now click on the **OK** button to complete the Mathcad installation. (You can leave the CD-ROM in the CD-ROM drive, ready for Stage 2 of the installation process, which follows shortly.)

The windows in Figure 1.5 may appear on your screen. If so, close them.

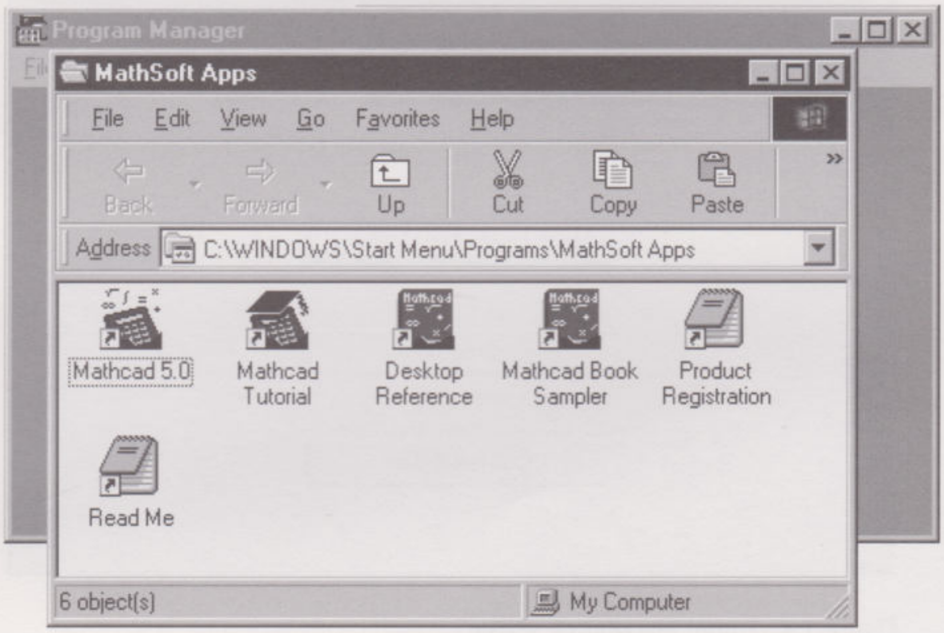


Figure 1.5 Unwanted windows

If you have successfully installed Mathcad, then you should be able to locate it as follows.

Activity 1.2 Locating Mathcad

(a) With the screen as in Figure 1.1, open the **Start** menu and move the mouse arrow up to highlight the **Programs** line. Leave the mouse arrow in this position, and a second menu will appear (see Figure 1.6). If the installation was successful, you should see **MathSoft Apps** as one item on the second menu. (The other items you see in this menu may be different from those shown here.)

(b) Move the mouse arrow over to highlight **MathSoft Apps**. After a short pause, a third menu will appear (see Figure 1.6). Here you should see the **Mathcad 5.0** icon (as well as icons for **Desktop Reference**, **Mathcad Book Sampler** and **Mathcad Tutorial**).

In future, this menu item will be referred to as the '**Mathcad 5.0** icon'.

(c) Once you have located the **Mathcad 5.0** icon, click on the **Start** button to close these menus.

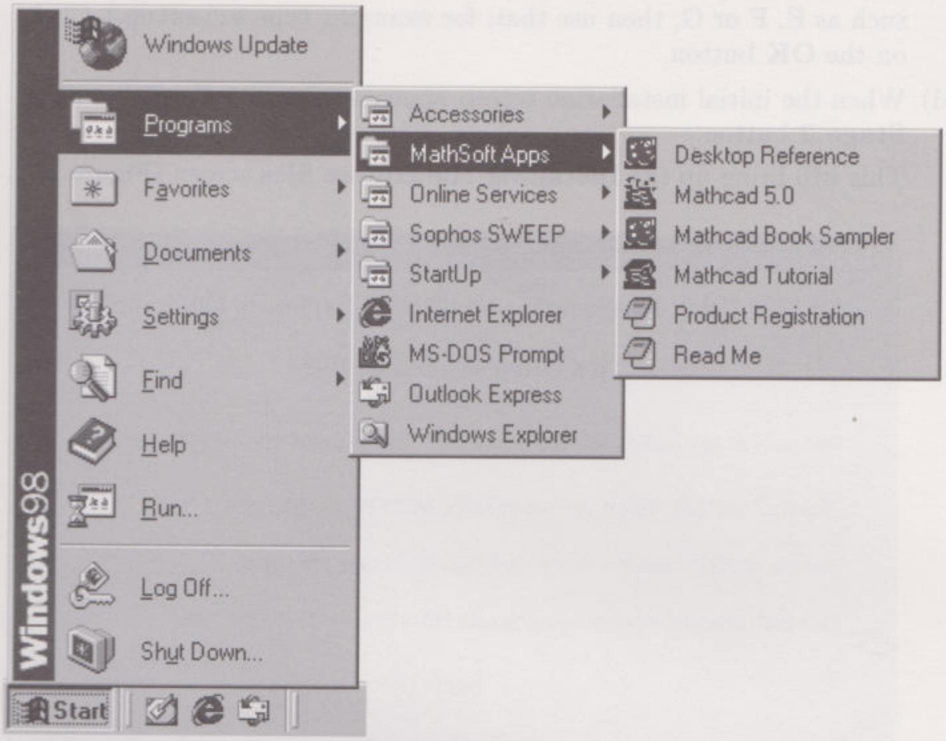


Figure 1.6 Locating Mathcad

Comment

When the time comes, you can run Mathcad by clicking on the **Mathcad 5.0** icon. (Do *not* do this now.)

The tutorial, desktop reference and book sampler come with the package and are not part of the MST121 or MS221 teaching materials. Though they contain some interesting material, which you may want to glance at later, you should *not* work through them now. Instead, proceed directly to Subsection 1.2.

If you do plan to look at the tutorial later, please consult the information about it in a Stop Press.

If you have successfully installed Mathcad, then you will now have located it (as in Figure 1.6). If Mathcad is not to be seen, then you should try re-installing it.

1.2 Installing the course files

You now need to use the MST121/MS221 CD-ROM again, in order to install the course files. This is the second (and final) stage of the installation process.

Activity 1.3 Installing the course files

First make sure that your screen looks something like Figure 1.1. If there are any applications running, then close them now.

(a) Check that the CD-ROM is in the CD-ROM drive.

(b) Open the **Start** menu, and select the **Run** option (Figure 1.2). You will see a dialog box similar to Figure 1.3, with the cursor flashing in the **Open** box.

(c) Type `d:\setup` in the box. (If your CD-ROM drive has another letter, such as E, F or G, then use that; for example, type `e:\setup`.) Click on the **OK** button.

(d) When the initial installation screen appears (Figure 1.4), click on the **Stage 2** button.

This will bring up the **Installing the course files** screen (Figure 1.7).

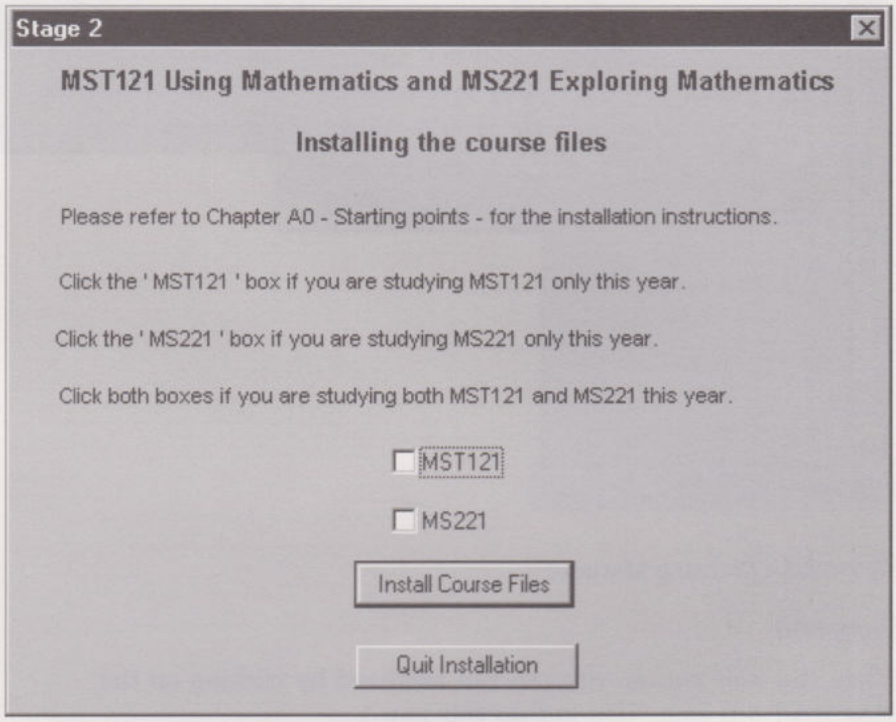


Figure 1.7 Stage 2 installation screen

(e) Select the course(s) that you are studying this year by clicking on the appropriate check box(es). (A selected check box is marked; clicking again on the box will cancel the selection and empty the box. Both check boxes can be selected.) Then click on the **Install Course Files** button.

(f) You will then be asked where on your computer you want to install the course files for the courses you have selected. You should accept the default directories offered, by clicking **OK**, unless you deliberately want to arrange things differently.

(g) The course files will now be installed automatically. A window saying **Installation was Successful** will appear when the installation is complete. Click on the **OK** button to finish.

(h) Now remove the CD-ROM from the drive and store it safely, away from heat, light and moisture.

(i) If you have installed the course files for MST121, then there will now be a window on the screen, named **MST121 Block D**, containing four icons named **CLT**, **OUStats**, **Simulations** and **StatsAid**; close this window.

Details of these programs and how to use them will be provided later, in the Computer Book for Block D.

All the MST121 course files should now have been placed in the directory **C:\Mst121**, and the directory structure on your hard disk will be similar to Figure 1.8. There are folders for most, but not all, of the MST121 chapters.

If you have installed the course files for MS221, then there will be a similar directory structure based on **C:\Ms221**.

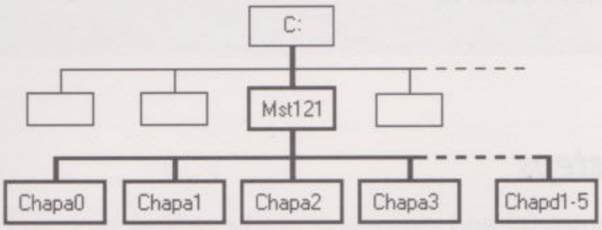


Figure 1.8 MST121 directory structure

This structure can be viewed using the *Windows* Explorer program (see Figure 1.9).

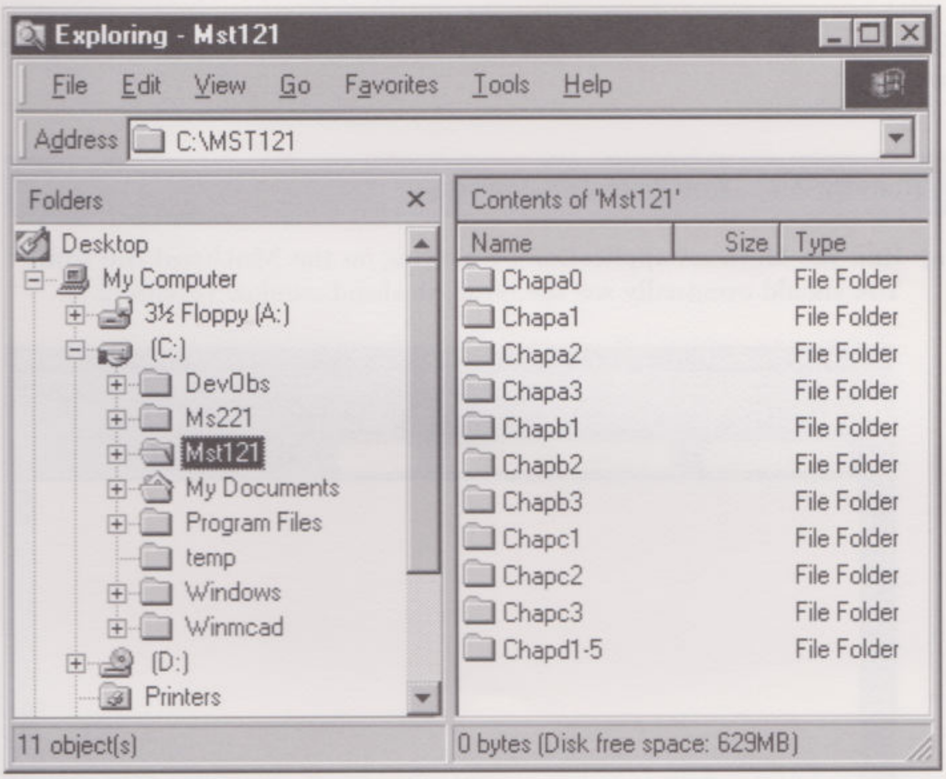


Figure 1.9 *Windows* Explorer program

If you look deeper, then you will find that the folder **Chapa0** contains five files; these are the introductory files that you will be using shortly. The other folders contain further Mathcad files and the statistics programs for Block D of MST121. The directory structure is necessarily highly organised and hierarchical in nature. As long as you keep to the suggested structure, you will be able to store and find files as needed throughout the course.

Everything should now be ready for you to start the introductory Mathcad sessions in Section 2. From now on you will use the copies of Mathcad and the course files on your hard disk. If anything goes wrong with Mathcad or these files during the activities, or during your own independent experimentation (which is thoroughly encouraged), then you can re-install them from the CD-ROM as described above.

The Mathcad installation password remains valid.

Summary of Section 1

In this section, you should have:

◇ installed Mathcad and located the icon used to run it;

◇ installed the course files.

2 Introduction to Mathcad

2.1 First steps

Mathcad is a powerful and expressive mathematical tool, and you will use it frequently throughout the course. It has many features, some of them quite technical, which will be introduced gradually in the context of teaching material. The rest of this chapter introduces some of the basic Mathcad facilities. You will also learn the conventions and working practice that the course adopts when using Mathcad.

If you have not continued straight on from Section 1, then switch on your computer.

Activity 2.1 Your first calculation

Locate the **Mathcad 5.0** icon as in Activity 1.2.

(a) Run the Mathcad application by clicking on the **Mathcad 5.0** icon. You should eventually see the basic Mathcad window (Figure 2.1).

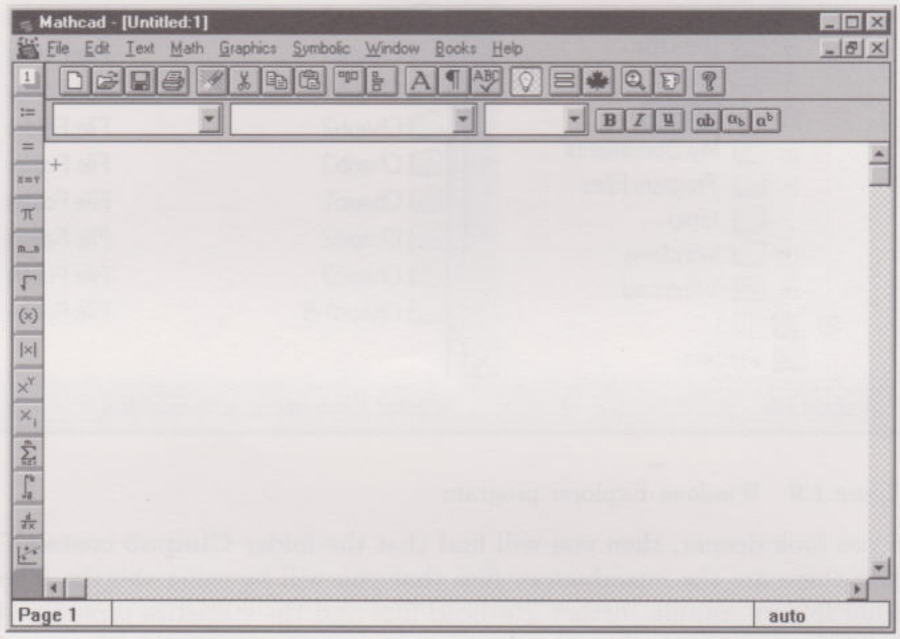


Figure 2.1 The Mathcad window

The maximise button is usually the middle one of the three buttons in the top right corner.

(b) If the Mathcad window does not fill your screen, use the maximise button in the top right corner to expand it.

(c) Mathcad automatically starts with a new, blank document. You will see a small red cross cursor (in the form of a cross-hair) on the screen, and the mouse arrow. (Depending on the size of your screen, you may or may not see solid and dotted vertical lines indicating the right-hand margin and edge of the page.) Move the mouse arrow to the middle of the window, and click the mouse button to move the red cross cursor to that point.

(d) Next you are asked to type in a calculation. You need to follow the typing instructions exactly. In particular, you *must not use the spacebar* to obtain extra spaces in an expression. They are not needed, and using the spacebar may have unwanted side-effects.

Now carefully type

```
131+285=
```

On the screen you will see

$$131 + 285 = 416$$

(e) The (vertical) blue bar cursor is still positioned after the number 285, so you can extend the calculation (for example, by adding another number). To finish the calculation, remove the blue bar cursor by moving the mouse arrow to a fresh point in the document and clicking the mouse button.

You can also finish the calculation by pressing the enter key [↵].

There is no need to keep this document containing a single calculation, so let's get rid of it.

Activity 2.2 Closing a document

(a) Click on **File** in the menu bar to see the following menu.

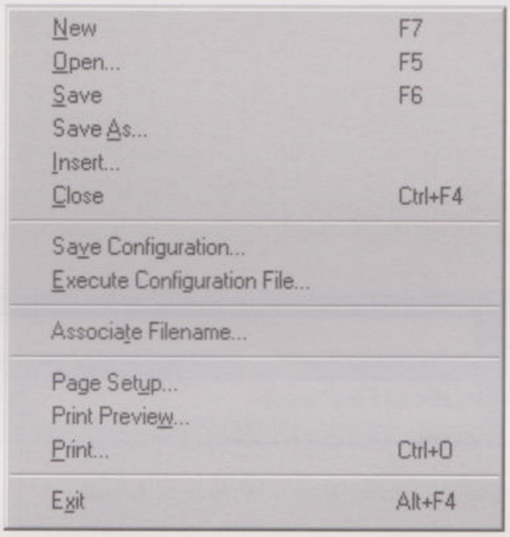


Figure 2.2 The **File** menu

(b) Select **Close**. You will see the dialog box in Figure 2.3.

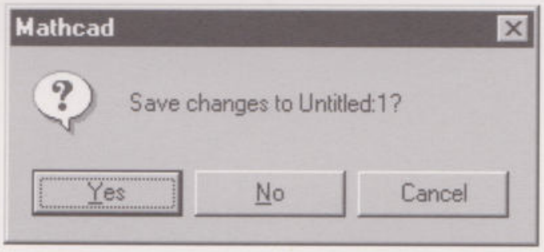


Figure 2.3 Save changes dialog box

(c) Click on the **No** button to get rid of the document without trace.

The next task is to create a new, blank document.

Activity 2.3 Creating a new document

(a) Click on the **File** menu, and select **New**.

Although you type ∗ for multiplication in Mathcad, multiplication appears on the screen as a centred dot.

(b) Now try some calculations of your own! Use +, −, asterisk ∗ (`[Shift]8`) for multiplication and the forward slash / for division.

To save the document created in Activity 2.3 as a file, proceed as follows.

Activity 2.4 Saving your own document

(a) Click on the **File** menu, and select **Save As**. You will see the Save As dialog box in Figure 2.4.

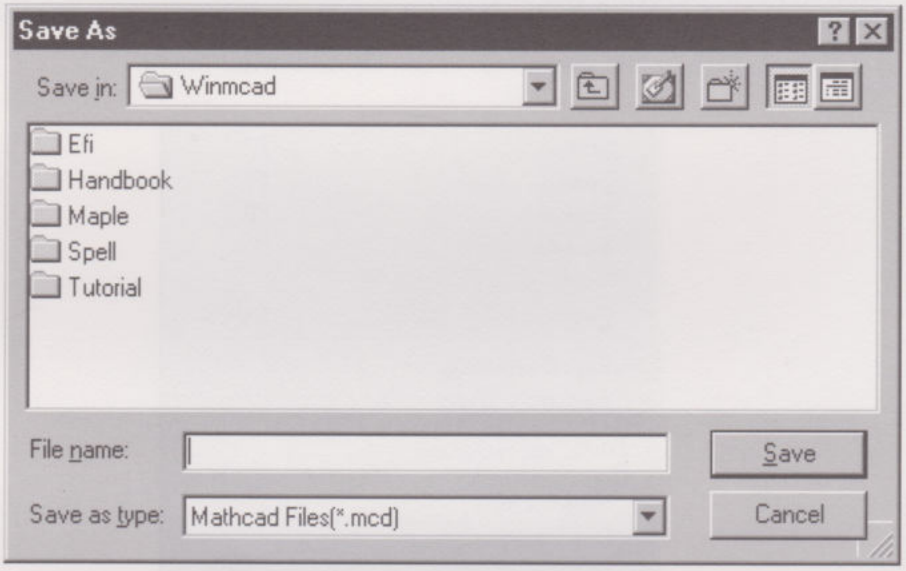


Figure 2.4 Save As dialog box – Winmcad folder

If you are studying MS221 only, without ever having done MST121, then locate the **Ms221** folder.

(b) Click on the drop-down arrow to the right of the **Save in** box. Click on the hard disk (**C:**) icon. Then use the scroll bar to locate the **Mst121** folder and double-click on it: the result is shown in Figure 2.5.

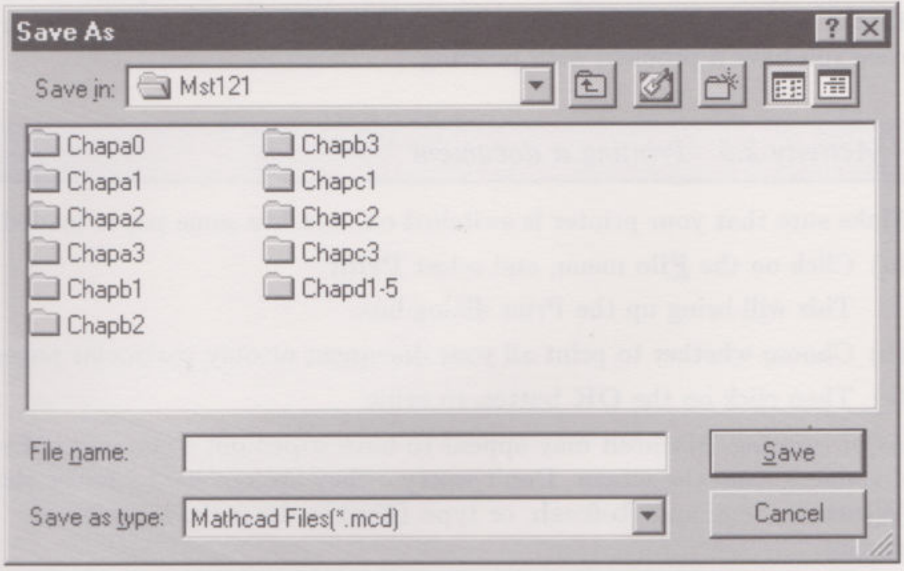


Figure 2.5 Save As dialog box – Mst121 folder

(c) Double-click on **Chapa0** to obtain the screen in Figure 2.6, and then type **my00001** (or any other acceptable name that you prefer) in the **File name** box. Finally, click on the **Save** button.

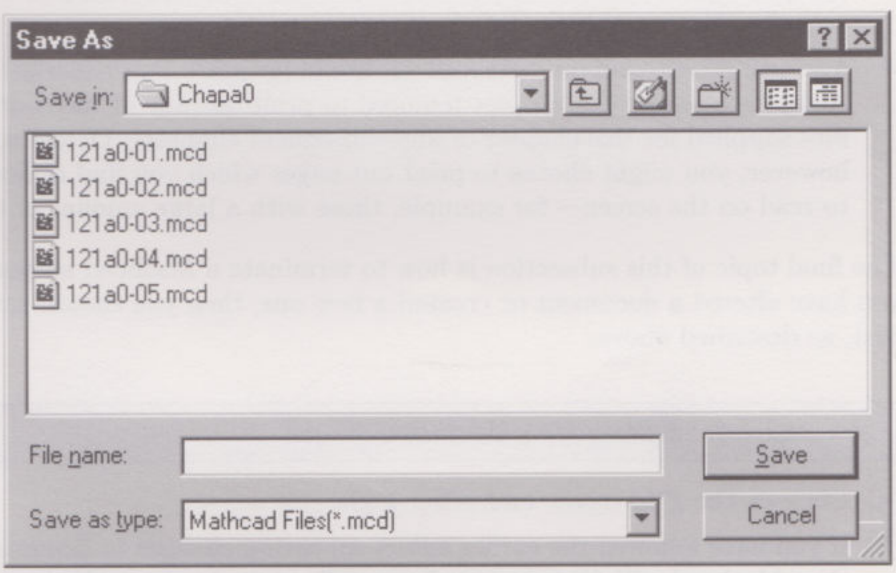


Figure 2.6 Save As dialog box – Chapa0 folder

Now that you have saved your document, you can go on to print it. Note that it is always good practice to save a document *before* printing it, in case you have a problem with printing.

Activity 2.5 Printing a document

Make sure that your printer is switched on, and has some paper loaded!

(a) Click on the **File** menu, and select **Print**.

This will bring up the Print dialog box.

(b) Choose whether to print all your document or only particular pages.

(c) Then click on the **OK** button to print.

After printing, Mathcad may appear to have wiped out some parts of your document from the screen. Don't worry – they are not lost! Choose the **Window** menu and **Refresh**, or type `[Ctrl]r`, to re-draw the screen display.

Hold down the control key, `[Ctrl]`, and then press `r`.

Please note the following points about printing.

Should you experience problems with printing from Mathcad, consult a Stop Press where up-to-date advice is given.

◇ It is very important to have a working printer set up in order to use Mathcad successfully. Mathcad needs printer information to set the correct margins and page layout for its documents on screen, irrespective of whether or not the printer is switched on and you actually print anything.

◇ You will need to supply Mathcad printouts for most Tutor-marked Assignments, but you are *not* required to print out any of the course files supplied for this chapter or any subsequent chapters. Occasionally, however, you might choose to print out pages which you find difficult to read on the screen – for example, those with a large amount of text.

The final topic of this subsection is how to terminate a Mathcad session. If you have altered a document or created a new one, then you should save it first, as described above.

Activity 2.6 Exiting from Mathcad

Even if you wish to move straight on to Subsection 2.2, it is a good idea to learn how to exit from Mathcad at this point.

(a) Click on the **File** menu, and select **Exit**.

(b) If you have followed the earlier advice on saving changes to documents, then Mathcad will disappear and you will return to the desktop.

If, however, you have made unsaved changes, then you will see the same dialog box as in Figure 2.3.

If you don't want to save the changes, click on the **No** button.

If you want to save the changes, click on the **Yes** button and follow the instructions in Activity 2.4.

You have now finished your first Mathcad session, and can close down the computer in the usual way, if you wish.

2.2 Documents, expressions and text

Now it is time for you to start working through the introductory course files. These contain detailed Mathcad documents with full comments. You should work through them carefully, as they are the basis for later work.

Begin the session by running Mathcad as described in Subsection 2.1.

Activity 2.7 Opening the first Mathcad file, 121a0-01.mcd

The file extension for Mathcad files is mcd.

Depending on your computer system, you may see the folder and file names displayed in a mixture of upper- and lower-case letters.

(a) Click on the **File** menu, and select **Open**. You will see something like the Open dialog box in Figure 2.7.

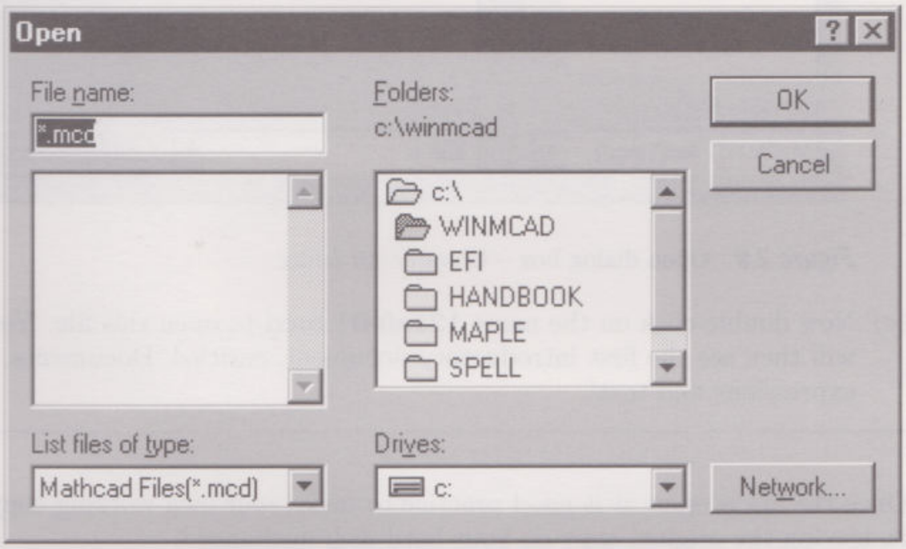


Figure 2.7 Open dialog box – Winmcad folder

(b) In the **Folders** box you should double-click on the hard disk **c:** folder, and then use the scroll bar to find the folder for the **Mst121** directory. Double-click on it. You will obtain a screen like Figure 2.8.

If you are studying MS221 only, without ever having done MST121, then find the folder for the **Ms221** directory, and work from that.

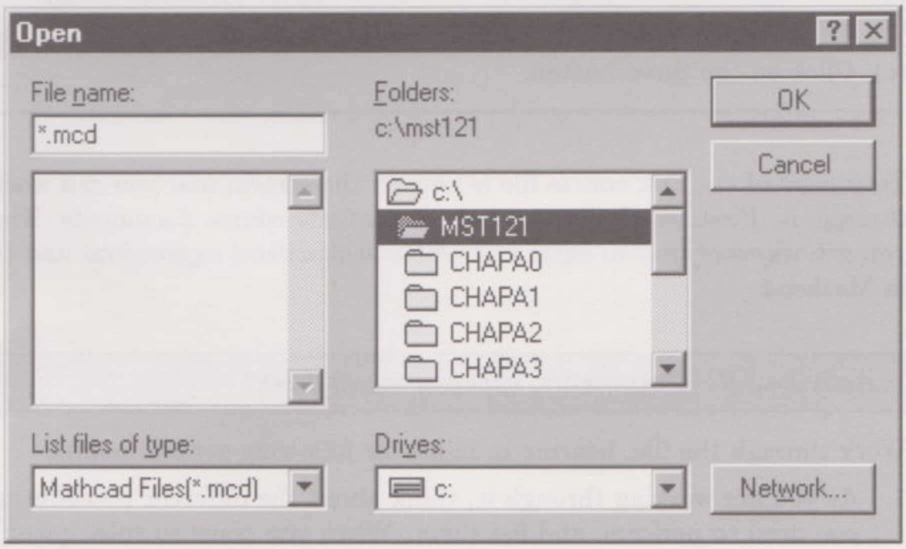


Figure 2.8 Open dialog box – Mst121 folder

Here you will see that the **Mst121** directory contains a closed directory for each chapter. Double-click on the folder for Chapter A0 (which contains the files for this chapter), and the **File name** box will change, as shown in Figure 2.9.

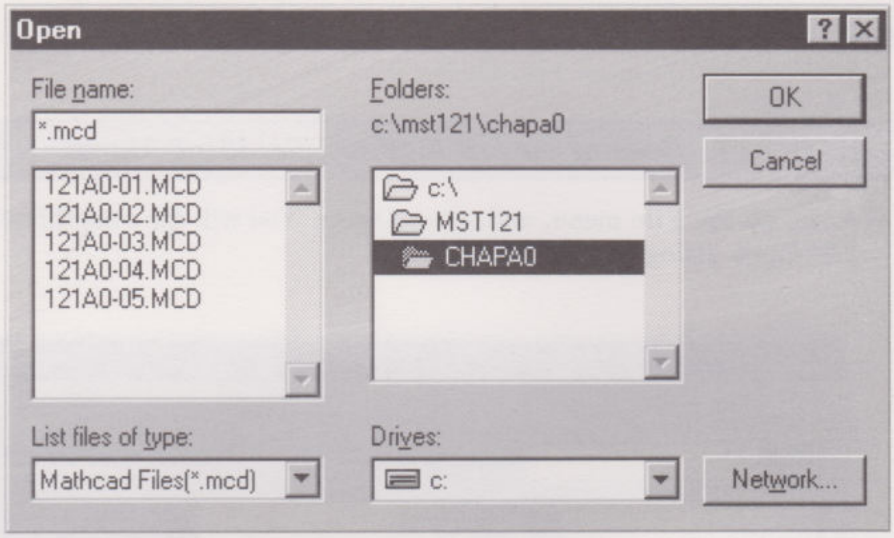


Figure 2.9 Open dialog box – Chapter A0 folder

(c) Now double-click on the name **121a0-01.mcd** to open this file. You will then see the first introductory document, entitled 'Documents, expressions and text'.

Once the file is open, it is good practice to make your own working copy of it, leaving the original copy on your hard disk unchanged.

Activity 2.8 Making your own working copy of a file

(a) Click on the **File** menu, and select **Save As**.
You will see the Save As dialog box, as in Figure 2.6.

(b) Now edit the **File name** box to contain **mya0-01.mcd** (or any acceptable new name that you prefer).

(c) Click on the **Save** button.

Your copy of the first course file is now on the screen, and you can work through it. First, you will learn how to navigate course documents. Then you will discover how to enter and edit mathematical expressions and text in Mathcad.

Activity 2.9 Documents, expressions and text

Work through the file, bearing in mind the following general points.

1. As you are working through it, think about the different procedures you need to perform, and list them. When you come to subsequent activities based on the computer, look at your list and see whether you want to make any changes to improve the way you tackle these activities.

2. There is quite a large amount of text in this and the other introductory Mathcad documents. If you find it difficult to read so much text on the screen, then you can print out some of the pages of a document (as described in Activity 2.5) and read through the printed version, returning to the screen to perform the various tasks in the document. Also, you may wish to print out some of the key pages giving Mathcad techniques, for later reference.

3. You can finish a computer session:

   ◇ when you have worked through a whole document;

   ◇ when you want a rest (at any time);

   ◇ when you want to work on something else.

   The steps for closing a document and exiting from Mathcad were given in Subsection 2.1.

4. When finishing, you are advised to save your *current* version of the Mathcad document, which will then incorporate any alterations you have made to it during the session. If you wish to return to this document, remember to open your copy of it, for example, **mya0-01.mcd**, unless you want to see the original version **121a0-01.mcd**.

The next introductory course file shows you, in more detail, how both mathematical expressions and text can be manipulated on the screen.

Activity 2.10 Rearranging documents

Locate in the **chapa0** folder the file **121a0-02.mcd**. Open it, save your own copy, and work through the document.

Summary of Section 2

In this section, you should have obtained a preliminary idea of Mathcad's features, which will be developed further as the course progresses. In particular, you have seen how to:

◇ run Mathcad and exit from Mathcad;

◇ create a new Mathcad document;

◇ save, print and close a Mathcad document;

◇ open a course file and make your own working copy of it;

◇ navigate a Mathcad document using the scroll bar, `[Shift][Page Up]`, `[Shift][Page Down]` and **Go to Page**;

◇ enter and edit simple mathematical expressions and text;

◇ select, re-position and copy regions containing mathematical expressions or text, and refresh the Mathcad screen.

3 Calculation

More detail on these topics can be found in the *Revision Pack.*

In this section, the main features of 'numbers' and 'calculation' are reviewed briefly, and you are shown how to calculate with Mathcad.

3.1 Arithmetic

The decimal system

The decimal system uses the **digits** $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ to represent numbers in terms of powers of 10. For example,

$$12.345 = 1 \times 10 + 2 \times 1 + 3 \times \frac{1}{10} + 4 \times \frac{1}{100} + 5 \times \frac{1}{1000};$$

that is,

$$12.345 = 1 \times 10^1 + 2 \times 10^0 + 3 \times 10^{-1} + 4 \times 10^{-2} + 5 \times 10^{-3}.$$

Remember that if a is non-zero, then

$$a^0 = 1,$$
$$a^{-1} = 1/a,$$
$$a^{-2} = 1/a^2,$$

and so on.

The position of a given digit in the number relative to the decimal point indicates the power of 10 which that digit multiplies. The positions to the right of the decimal point are referred to as 'decimal places'; for example, in 12.345, the digit 5 is in the third decimal place.

The above number is a **finite decimal**; that is, it has only finitely many decimal places. Most calculations are carried out with finite decimals, but there are many quantities whose size cannot be described *exactly* with finite decimals. For example, the decimal representations of $1/3$ and $\sqrt{2}$, namely $1/3 = 0.333\ldots$ and $\sqrt{2} = 1.414\ldots$, both 'go on forever'. Therefore we also need types of numbers whose decimal representations are **infinite decimals**; such decimals have infinitely many decimal places.

The table below classifies various types of numbers, with examples of each. In the column on the left are the names of these types of numbers (with various alternative names in brackets), and the mathematical symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ used to denote the corresponding sets of numbers. For example, $\mathbb{N}$ denotes the set of all natural numbers: $1, 2, 3, \ldots$. (When writing these symbols, just do the best you can!)

The symbol '...' means 'and so on'. It indicates that something has been omitted whose nature can be deduced from the context.

Table 3.1 Types of numbers

Number type	Examples
natural numbers, $\mathbb{N}$ (counting numbers, positive integers)	$1, 2, 3$
integers, $\mathbb{Z}$ (whole numbers)	$-3, -2, -1, 0, 1, 2, 3$
rational numbers, $\mathbb{Q}$ (fractions)	$\frac{1}{2} = 0.5,\ \frac{1}{3} = 0.333\ldots,\ \frac{3}{1} = 3,\ -\frac{9}{11} = -0.8181\ldots$
real numbers, $\mathbb{R}$ (decimals – finite and infinite)	$\frac{1}{2} = 0.5,\ \frac{1}{3} = 0.333\ldots,\ \sqrt{2} = 1.4142\ldots,\ \pi = 3.1415\ldots$

There follow some observations about this table.

◇ The four types of number in the table are increasingly general: every natural number is an integer, every integer is a rational number, and every rational number is a real number.

◇ The symbol $\mathbb{Z}$ derives from the German word *Zahl*, meaning 'number', $\mathbb{Q}$ derives from the word 'quotient', and $\mathbb{R}$ derives from the word 'real'.

The word 'real' distinguishes these numbers from other numbers – for example, complex numbers, studied in MS221 – which are sometimes considered to be 'unreal'.

◇ A **recurring decimal** is one for which a particular digit or block of digits is repeated endlessly to the right. For example: $\frac{1}{3} = 0.333\ldots$ and $\frac{29}{54} = 0.5370370\ldots$. Two notations in common use for recurring decimals are indicated below:

$$\tfrac{1}{3} = 0.\dot{3} = 0.\overline{3} \quad \text{and} \quad \tfrac{29}{54} = 0.5\dot{3}7\dot{0} = 0.5\overline{370}.$$

These recurring decimal representations can be found by long division or with a calculator (up to the accuracy of the calculator).

This course uses the dot notation for recurring decimals.

◇ In fact, all rational numbers have finite or recurring decimal representations, and all numbers with finite or recurring decimal representations are rational. Also, finite decimals may be considered as recurring, since we can include the so-called 'trailing zeros' which are usually omitted. For example: $0.5 = 0.5000\ldots$.

◇ Real numbers that are not rational are called **irrational numbers**; these are the ones with non-recurring decimal representations. For example, it can be shown that $\sqrt{2}$ and π are both irrational.

You can see from these examples that numbers can be represented in more than one way. For example, $\frac{1}{2} = 0.5$. Therefore, when we perform calculations, we may have to decide which of these representations to use. For example, faced with $\frac{1}{3} + \frac{1}{6}$, it is more straightforward to use the rules for fractions,

$$\tfrac{1}{3} + \tfrac{1}{6} = \tfrac{2}{6} + \tfrac{1}{6} = \tfrac{3}{6} = \tfrac{1}{2},$$

rather than adding the infinite decimals,

$$\tfrac{1}{3} + \tfrac{1}{6} = 0.333\ldots + 0.1666\ldots.$$

$\frac{1}{3}$ and $\frac{2}{6}$ are equivalent fractions. Fractions with the same denominator, like $\frac{2}{6}$ and $\frac{1}{6}$, are added by adding the numerators.

The real line

The various representations of numbers each have their own advantages and disadvantages. For example, the decimal representation is convenient when we wish to locate real numbers on a number line (such as the x-axis of a graph). On such a line, the integers are represented by equally spaced points, appropriately labelled and increasing from left to right, as in Figure 3.1.

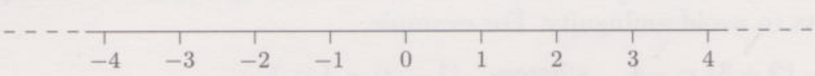


Figure 3.1 A number line

The segment of the number line between each adjacent pair of integers is divided into 10 equal segments, the division points being labelled with corresponding tenths. Each of these 10 segments is then itself divided into 10 equal segments, the division points being labelled with corresponding hundredths. This is illustrated in Figure 3.2.

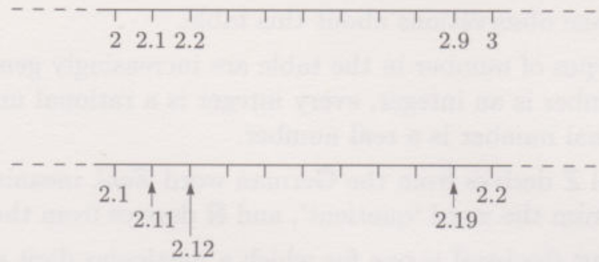


Figure 3.2 Dividing into ten segments

We imagine this process being repeated indefinitely. Then any finite decimal can be located, at least in principle, on the number line. For example, 1.414 lies between 1.41 and 1.42, four tenths of the way along.

Even *infinite* decimals can be located in this way. For example, the unique point corresponding to $\frac{1}{3} = 0.333\ldots$ lies to the right of each of the finite decimals

$$0,\ 0.3,\ 0.33,\ 0.333,\ 0.3333,\ \ldots,$$

and to the left of each of the finite decimals

$$1,\ 0.4,\ 0.34,\ 0.334,\ 0.3334,\ \ldots.$$

In a similar way, you can locate $\sqrt{2} = 1.414\ldots$ on the number line.

We refer to a line which represents all possible real numbers in this way as the **real line**.

Decimal representation facilitates the comparison of real numbers. For example, the numbers $\sqrt{2}$ and $10/7$ have decimal representations $\sqrt{2} = 1.414\ldots$ and $10/7 = 1.428\ldots$. These differ first in the second decimal place. This indicates that $\sqrt{2}$ is less than $10/7$, and so lies to the left of $10/7$ on the real line. We write '$\sqrt{2}$ is less than $10/7$' in symbols as

$$\sqrt{2} < 10/7 \quad \text{or, equivalently,} \quad 10/7 > \sqrt{2}.$$

The latter form is read as '$10/7$ is greater than $\sqrt{2}$'.

Similarly, the symbols $\leq$ and $\geq$ are read as 'less than or equal to' and 'greater than or equal to', respectively. Thus $6 \leq 6$ and $6 \leq 7$ are both true statements.

Arithmetic and rounding

The four basic arithmetic operations,

$$+\ (\text{addition}), \quad -\ (\text{subtraction}), \quad \times\ (\text{multiplication}), \quad \div\ (\text{division}),$$

can be applied to pairs of real numbers to yield a real number as the answer – the only exception to this is 'division by 0', which is not defined. For example,

$$1 + 2 = 3, \quad 3 - 4 = -1, \quad 5 \times (-6) = -30, \quad 7 \div 8 = 0.875.$$

Alternative notations are

$$5 \times (-6) = 5 \cdot (-6),$$
$$7 \div 8 = 7/8 = \tfrac{7}{8}.$$

Calculations involving more than one of the basic operations often require brackets to avoid ambiguity. For example,

$$1 - (2 + 3) = -4, \quad \text{whereas} \quad (1 - 2) + 3 = 2.$$

The above calculations can all be performed by hand, without help from a calculator or computer, and the results are exact. But many calculations in which we want the answer as a decimal are most conveniently done with a calculator or computer, and in such cases the answer presented is often not exact. For example, using a calculator, we obtain

$$\frac{98}{78} = 1.256\,410\,256\,4 \quad (\text{to 10 d.p.}).$$

$\frac{98}{78}$ has a recurring decimal representation.

This answer is not exact, but has been rounded to ten *decimal places* (d.p.); that is, the answer presented is the number with ten decimal places

which is nearest to the exact answer. Recall that we round to a given number of decimal places by keeping the digits up to that decimal place, and if the first discarded digit is 5 or more, increasing the last digit kept by 1. For example,

$$1.364 = 1.36 \text{ (to 2 d.p.)} \quad \text{and} \quad 1.367 = 1.37 \text{ (to 2 d.p.)}.$$

Different calculators may round to different numbers of decimal places. For example, on a calculator which rounds to eight decimal places, we would obtain

$$\frac{98}{78} = 1.256\,410\,26 \quad \text{(to 8 d.p.)},$$

since the digit in the 9th decimal place of 1.256 410 256 4 is 6. You will see shortly that Mathcad allows you to *choose* the number of decimal places to which displayed answers are rounded.

We often choose to round answers correct to a particular number of *significant figures* (s.f.), rather than to a particular number of decimal places. This means that the answer presented is the number with the given number of digits (counting from the first non-zero digit) which is nearest to the exact answer. (The rule for the last digit is the same as when rounding to a number of decimal places.) For example,

The Appendix contains guidelines on how to present numerical answers to questions.

$$\frac{98}{78} = 1.2564 \text{ (to 5 s.f.)} \quad \text{and} \quad \frac{82}{7} = 11.714 \text{ (to 5 s.f.)}.$$

Powers

Another situation in which we often need help from a calculator or computer is in calculating powers.

Other words for 'power' are 'exponent' and 'index'.

First, here is a reminder of some of the rules for handling *integer* powers. If a is a non-zero real number, and p and q are integers, then

$$a^p \times a^q = a^{p+q} \quad \text{and} \quad (a^p)^q = a^{pq}. \tag{3.1}$$

For example, with $a = 2$, $p = 3$ and $q = -4$,

$$2^3 \times 2^{-4} = 2^{3-4} = 2^{-1} = \tfrac{1}{2} \quad \text{and} \quad \left(2^3\right)^{-4} = 2^{3\times(-4)} = 2^{-12} = \tfrac{1}{4096}.$$

Next, more general powers are discussed. Suppose that a is a *positive* number. If n is a positive integer, then $\sqrt[n]{a}$ denotes that *positive* number which, when raised to the power n, gives the answer a; it is called the **nth root** of a. For example, $\sqrt[3]{8} = 2$ because $2^3 = 8$. Also recall that for any positive integer n, $\sqrt[n]{0} = 0$. Most roots cannot be found easily by hand calculation, however.

For $n = 2$, this is the **square root** of a, written $\sqrt{a}$.

A more convenient notation for the nth root is $a^{1/n}$. This notation has two important advantages.

◇ It is consistent with the rules for integer powers stated in equation (3.1). For example, on using the second of these rules, with $a > 0$, $p = 1/n$ and $q = n$, we obtain

$$\left(a^{1/n}\right)^n = a^{(1/n)\times n} = a^1 = a,$$

as expected.

◇ It can be extended to allow any power of a *positive* number a. First, if m and n are integers, with n positive, then

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m \quad \text{or} \quad a^{m/n} = \sqrt[n]{a^m}.$$

These two definitions give the same value for $a^{m/n}$.

For example,

$$8^{4/3} = \left(\sqrt[3]{8}\right)^4 = 2^4 = 16 \quad \text{or} \quad 8^{4/3} = \sqrt[3]{8^4} = \sqrt[3]{4096} = 16.$$

This extension is covered in courses on real analysis.

This defines a^x when x is a rational number m/n. The extension to values of x which are arbitrary real numbers (such as $2^{\sqrt{2}}$, which corresponds to $a = 2$ and $x = \sqrt{2}$) can be carried out by using approximations to x by rational numbers.

With these extensions, the rules for powers in equation (3.1) continue to hold, as do rules such as $\sqrt{ab} = \sqrt{a}\sqrt{b}$.

Scientific notation

Before you try some calculations involving powers, it is important to recall a variation of decimal notation which is used for dealing with numbers which are either very large or very small. This is **scientific notation** – also known as **exponential notation** and **standard form**. Scientific notation takes the following form:

(a number between 1 and 10, not including 10) × (a power of 10).

An alternative notation using E is illustrated below:

253 = 2.53E2,

0.253 = 2.53E–1.

For example,

$$253 = 2.53 \times 10^2, \quad 0.253 = 2.53 \times 10^{-1}.$$

The appropriate power of 10 is found by counting how many positions the decimal point has to be moved to obtain the required 'number between 1 and 10'. The sign of that power is positive if the decimal point is moved to the left, and negative otherwise. Different calculators choose different thresholds at which they change from displaying an answer in ordinary decimal notation to scientific notation. Mathcad allows you to *choose* this threshold.

Calculation with and without Mathcad

In the next two activities, you are invited to do some calculations by hand or with your calculator, and then with Mathcad.

Activity 3.1 Arithmetic and rounding

Use hand calculation or your calculator, as appropriate, to check that the following answers are correct.

(a) $625^{3/4} = 125$

(b) $78/98 = 0.795\,918$ (to 6 d.p.)

(c) $2^{4/3} = 2.520$ (to 3 d.p.)

(d) $2^{100} = 1.27 \times 10^{30}$ (to 3 s.f.)

In rounded numbers trailing zeros are sometimes omitted. So part (c) could be written as

$2^{4/3} = 2.52$ (to 3 d.p.).

Solutions are given on page 43.

The next activity leads you through the process of entering arithmetic expressions in Mathcad and evaluating them. You will also see how to choose the number of decimal places that Mathcad uses to display numerical answers, and the threshold for its use of scientific notation.

Switch on your computer and run Mathcad.

Activity 3.2 Calculating with Mathcad

Locate in the **chapa0** folder the file **121a0-03.mcd**. Open it, save your own copy, and work through the document.

Comment

This activity should convince you that for a single calculation, it is usually easier to use hand calculation or your calculator. Mathcad will be more effective when you are faced with many calculations of a similar type, and will also enable you to retain a record of your calculation.

3.2 Variables

A powerful feature of mathematics is its use of symbols (usually letters or combinations of letters) to represent numbers. We use symbols to represent numbers that may take various values; that is, they may vary. Such a symbol is called a **variable**.

In the discussion of powers, the variables a, p, q, m, n and x were used.
In particular, initially the variable a could be any non-zero real number. Then it was restricted to be only a positive number.

Variables are often related to other variables. For example, the formula for the circumference of a circle of a given radius is

$$C = 2\pi r.$$

Here r is a variable that denotes the radius of the circle, and C is a variable that denotes the circumference of the circle. The variables r and C can take only positive real values. The number $\pi = 3.141\,592\,6\ldots$ does not vary, so it is not a variable. Given any positive radius r, the formula enables you to calculate the corresponding circumference C. Since r can take *any* positive value, it is called an **independent variable** in this formula. On the other hand, C is a **dependent variable**, since its value depends on that of r. Similarly, in the formula for the area of a triangle

In Mathcad and on your calculator, the number π is a built-in constant.

$$A = \tfrac{1}{2}bh,$$

the independent variables b and h represent the lengths of the base and height of the triangle, and the dependent variable A represents the triangle's area.

The next two activities ask you to evaluate several formulas for particular values of the independent variables, and show you how to work with variables in Mathcad.

Activity 3.3 Using variables

Use your calculator to find the values of the dependent variables in each of the following formulas. Give your answers correct to three significant figures.

(a) The circumference C of a circle of radius r:

$$C = 2\pi r, \quad \text{for } r = 2 \text{ and } r = 5.$$

(b) The area A of a circle of radius r:

$$A = \pi r^2, \quad \text{for } r = 1.5 \text{ and } r = 3.7.$$

(c) The volume V of a cylinder of height h whose ends are circles of radius r:

$$V = \pi r^2 h, \quad \text{for } r = 1.5,\ h = 8.$$

(d) The hypotenuse c of a right-angled triangle with sides a, b, c:

$$c = \sqrt{a^2 + b^2}, \quad \text{for } a = 1, b = 2 \text{ and } a = 3, b = 4.$$

Solutions are given on page 43.

Switch on your computer and run Mathcad (if necessary).

Activity 3.4 Using variables in Mathcad

Locate in the **chapa0** folder the file **121a0-04.mcd**. Open it, save your own copy, and work through the document.

Comment

This activity should convince you that if you need to calculate the value of a dependent variable for *many* values of the independent variables, then using Mathcad is very efficient.

The earlier Mathcad documents in this chapter used the = sign to evaluate arithmetic expressions numerically. This document introduced the symbol := which is used in Mathcad to define variables and assign values to them. For example, the Mathcad expression $a := 3$ assigns the value 3 to the variable a. When *writing* mathematics, this appears as a sentence such as 'Now define $a = 3$.'.

Summary of Section 3

This section recalled basic properties of numbers and arithmetic, introduced some new notation for several types of numbers, and showed you how arithmetic is done with Mathcad. In particular, you met:

◇ the set of *natural* numbers, $\mathbb{N}$: $1, 2, 3, \ldots$;

◇ the set of *integers*, $\mathbb{Z}$: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$;

◇ the set of *rational* numbers, $\mathbb{Q}$: all numbers of the form m/n, where m, n are integers and $n \neq 0$;

◇ the set of *real* numbers, $\mathbb{R}$: all finite or infinite decimals, which together comprise the *real line*;

◇ formulas giving the value of a dependent variable in terms of independent variables;

and saw:

◇ that rational numbers are equivalent to *finite* or *recurring* decimals;

◇ rules which apply to powers and the meaning of rational powers;

◇ rounding numbers and scientific notation;

◇ how to enter and evaluate arithmetic expressions in Mathcad.

Exercises for Section 3

Exercise 3.1

(a) Use hand calculation or your calculator, as appropriate, to evaluate the following expressions, giving your answers to three significant figures.

(i) $49^{3/2}$ (ii) $50^{3/2}$ (iii) $\dfrac{26}{65}$ (iv) $\dfrac{27}{65}$ (v) 3^{-10}

(b) Use Mathcad to confirm your answers to part (a).

To use Mathcad, you will need to create a new, blank document.

Exercise 3.2

(a) Use your calculator to find the values of the dependent variables in each of the following formulas, giving your answers to three significant figures.

(i) The volume V of a sphere of radius r:

$$V = \tfrac{4}{3}\pi r^3, \quad \text{for } r = 2.5 \text{ and } r = 4.7.$$

(ii) The semi-perimeter s and area A of a triangle with sides a, b and c:

$$s = \tfrac{1}{2}(a+b+c) \quad \text{and} \quad A = \sqrt{s(s-a)(s-b)(s-c)},$$

for $a = 2$, $b = 3$, $c = 4$.

(b) Use Mathcad to confirm your answers to part (a)(i).

4 Algebra

The word 'algebra' comes from the Arabic *al-jabr*, meaning 'the reunion of broken parts'. Its usage stems from its inclusion in the title of a book by Mohammed bin Mûsâ al-Khowârizmî, written in Baghdad circa 825 AD.

When variables are used in problems, we obtain expressions and equations which involve these variables, linked together by precisely the same operations that are used with numbers; see, for example, the equations in Activity 3.3. Such expressions and equations can often be rearranged in useful ways to give equivalent expressions and equations. This process of rearrangement is called **algebra**.

Algebra has been crucial to the development of mathematics throughout most of its history. It permits mathematical statements to be expressed in concise and general ways. What is more, one algebraic statement may lead on to another, which is equivalent to the first, but provides a different emphasis and hence permits a new understanding of the underlying situation.

More detail on these topics can be found in the *Revision Pack*.

In this section, the basics of algebra are reviewed, and you are shown how to do many basic algebraic operations with Mathcad.

4.1 Algebraic expressions

An **algebraic expression** is a single number, a single variable or a collection of numbers and variables which are connected together by the basic arithmetic operations $+$, $-$, $\times$, $\div$, and also by powers. For example, all of the following are (algebraic) expressions:

$$2ax - 3by + z, \quad 5x^2 - 3x - 2, \quad \frac{4}{3(x+2)} + \frac{2}{x+3}, \quad \sqrt{x^2 + y^2}.$$

You saw in Activity 3.4 that in Mathcad we do not omit the multiplication sign in this way.

In such expressions, we usually omit the multiplication sign $\times$ between two expressions unless these are both numbers, or unless we wish to emphasise that multiplication is being used. Also, as with numbers, the division sign $\div$ is usually replaced by / or by a horizontal bar, and brackets are used to indicate parts of expressions which are to be evaluated together.

An expression often involves simpler expressions, known informally as **terms** of the original expression. For example, in the expression

$$2ax - 3by + z,$$

the expressions $2ax$, $-3by$ and z are terms.

An expression in which a number of terms are added together is called a **sum**. For example,

$$5x^2 - 3x - 2$$

is a sum with three terms, namely $5x^2$, $-3x$ and -2. The term -2 is a constant term.

An expression in which a number of terms are multiplied together is called a **product**, and each of the terms is a **factor** of the product. If a particular term of a product is of interest, then the remainder of the product is called the **coefficient** of that term. For example, $2a$ is the coefficient of x in the product $2ax$.

An algebraic expression in which one expression, the **numerator**, is divided by another expression, the **denominator**, is called a **fraction**, or **quotient**. For example, the fraction

$$\frac{4}{3(x+2)}$$

has numerator 4 and denominator $3(x+2)$.

Equivalent algebraic expressions

Two algebraic expressions are **equivalent** if they take the same value for all possible values of the variables; this equivalence is indicated using an = sign. There are several basic ways of rearranging an algebraic expression to obtain equivalent expressions. Some of these methods aim to simplify the expression, whereas others aim to put the expression into a form which makes it more convenient for a particular technique to be applied.

Collecting like terms A basic way to simplify a sum (or a product) is to collect together terms of the same type. For example, the sum

$$x^2 + 2x + 1 + x^2 - 2x + 1$$

has terms in x^2 and x, and constant terms. Collecting together like terms, we obtain

$$x^2 + 2x + 1 + x^2 - 2x + 1 = 2x^2 + 2.$$

Here the terms $2x$ and $-2x$ cancel out.

Sums like these, which involve powers of a variable, are usually arranged with the powers in increasing or decreasing order. For example, we would normally rearrange $2x + x^2 - 3$ as $x^2 + 2x - 3$ or as $-3 + 2x + x^2$. There is no 'correct' order in which to write a sum (or product), but experience often suggests which form is preferable in a particular context.

Cancelling common factors A basic way to simplify a fraction is to cancel any (non-zero) *factor* which appears in both the numerator and the denominator. For example,

$$\frac{60a^2b}{35a^5c} = \frac{12b}{7a^3c},$$

$a^5 = a^3 \times a^2$,
$60 = 12 \times 5$,
$35 = 7 \times 5$.

since the factors 5 and a^2 appear in the numerator and in the denominator.

Multiplying out brackets Two sums which are to be multiplied together are each placed within brackets. To find the product, we multiply each term of the first sum by each term of the second, then collect like terms of the resulting sum. For example,

This process is also called 'Expanding brackets'.

$$\begin{aligned}(a-b)(a+b) &= a^2 + ab - ba - b^2 \\ &= a^2 - b^2,\end{aligned}$$

Note that $ab = ba$.

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a^2 + ab + ba + b^2 \\ &= a^2 + 2ab + b^2.\end{aligned}$$

Factorising a sum Factorising an *integer* means expressing it as a product of smaller integers. For example, $18 = 3 \times 6$. Factorising a *sum* means expressing it as a product of simpler sums. This process reverses the effect of multiplying out brackets. For example, as you saw above,

$a^2 - b^2$ can be factorised as $(a-b)(a+b)$,
$a^2 + 2ab + b^2$ can be factorised as $(a+b)^2$.

This first factorisation is called the **difference of two squares**.

Finding a factor is relatively straightforward if all terms of the sum have an identifiable common factor which can be 'taken out'. For example,

$$x^3 + 2x^2 - 3x = x(x^2 + 2x - 3).$$

Later in this section, we consider the factorisation of expressions such as $x^2 + 2x - 3$.

Combining fractions The rules for combining algebraic fractions are the same as those for combining numerical fractions.

◇ To multiply fractions, multiply the numerators and multiply the denominators. For example,

$$\left(\frac{a}{2x}\right)\left(\frac{b}{3y}\right) = \frac{ab}{6xy}.$$

Turning a fraction upside down gives the fraction's **reciprocal**.

◇ To divide by a fraction, turn it upside down and multiply. For example,

$$\left(\frac{a}{2x}\right) \div \left(\frac{b}{3y}\right) = \left(\frac{a}{2x}\right)\left(\frac{3y}{b}\right) = \frac{3ay}{2bx}.$$

◇ To add (or subtract) fractions, first arrange that the fractions have a common denominator, then add (or subtract) the numerators. The simplest way to obtain a common denominator is to multiply the various denominators together. In the following example, a common denominator is $(2x)(3y) = 6xy$:

Usually, the first step in this rearrangement is not explicitly written down. With practice, it is possible to write down the final step immediately.

$$\begin{aligned}\frac{a}{2x} - \frac{b}{3y} &= \left(\frac{a}{2x}\right)\left(\frac{3y}{3y}\right) - \left(\frac{b}{3y}\right)\left(\frac{2x}{2x}\right)\\ &= \frac{3ay}{6xy} - \frac{2bx}{6xy}\\ &= \frac{3ay - 2bx}{6xy}.\end{aligned}$$

Expanding a fraction If the numerator of a fraction is a sum, then the fraction can be written as a sum of simpler fractions. For example,

$$\frac{x^2 + 2x + 3}{x^2 + 1} = \frac{x^2}{x^2 + 1} + \frac{2x}{x^2 + 1} + \frac{3}{x^2 + 1}.$$

The expression on the left has been 'expanded' to obtain the longer expression on the right.

The next two activities ask you to practise some of these methods of rearranging expressions, and show you how they can be performed in Mathcad.

Activity 4.1 Rearranging algebraic expressions

(a) Collect like terms in the expression

$$(a - 2b + 3c - 4d) - (-4a + 3b - 2c + d).$$

(b) Cancel common factors in the fraction

$$\frac{20a^2b}{15ab^2}.$$

(c) Multiply out each of the following products.

(i) $(x + 1)(x - 2)$ (ii) $(a - b)^2$ (iii) $(t + 1)(t + 2)(t + 3)$

(d) Factorise each of the following expressions.

(i) $a^2b - ab^2$ (ii) $x^2 - 4$ (iii) $x^2 - 2x + 1$

(e) Add the following fractions, giving the numerator and denominator in factorised form:

$$\frac{4}{3(x+2)} + \frac{2}{x+3}.$$

(f) Expand the following fraction, simplifying each fraction of the resulting sum:

$$\frac{x^2+2x+1}{x^3}.$$

Solutions are given on page 43.

Switch on your computer and run Mathcad.

Activity 4.2 Rearranging algebraic expressions in Mathcad

Locate in the **chapa0** folder the file **121a0-05.mcd**. Open it, save your own copy, and work through the document.

Comment

This activity shows that Mathcad can rearrange algebraic expressions, though it may produce answers in unexpected forms. For the examples in Activity 4.1, there is no reason to use Mathcad, since these can all be done readily by hand. For more complicated examples, such as multiplying out $(x-1)(x-2)(x-3)(x-4)$, Mathcad is useful (if only as a check on your working!).

4.2 Solving equations

An **equation** is any statement in which two expressions are placed on either side of an = sign to indicate that they are equal. Equations may be of several types, as follows.

- ◇ As you have seen, an equation may indicate that two expressions are equivalent. For example, the product $3 \times 6 = 18$ and the factorisation $a^2 - b^2 = (a-b)(a+b)$ are of this type. Such an equation is sometimes called an **identity**.
- ◇ An equation may define a dependent variable in terms of one or more independent variables. An example is the equation $A = \pi r^2$, which defines the value of the variable A. Such an equation is often called a **formula**, and the variable defined is called the **subject** of the formula.
- ◇ An equation may represent a **condition** which is present in a particular problem. For example, suppose that a number x has the property that if we double x and then add 1, then the answer is 85. This property can be expressed by the equation $2x + 1 = 85$. Here, the aim is to find the value of x; such a variable x is called an **unknown**. In an equation with several variables, we often wish to express one unknown variable in terms of the others, as a formula.

The symbol ≡ (instead of =) is sometimes used for an identity, but not in this course.

You saw in Activity 3.4 that Mathcad uses the symbol := to define the value of a variable.

The letters x, y and z are often used for unknown variables, with a, b, c, and so on, for others. There is no 'rule' about this, however!

It is usually clear from the context which meaning a given = sign has. When reading and writing algebra, it will help if you are conscious of these different meanings.

Two equations are called **equivalent** if one can be obtained from the other by successively changing the equation in the following ways:

◇ applying any of the operations $+$, $-$, $\times$, $\div$ in the same way to both sides of the equation (avoiding multiplication or division by 0);

◇ replacing any of the expressions in the equation by equivalent expressions;

◇ exchanging the left-hand and right-hand sides.

Solving an equation (or equations) means finding all the values of the unknown (or unknowns). Such a value is called a **solution** of the equation.

Linear equations

A **linear expression** is a sum, such as $2x + 3y + 1$, consisting of first powers of the variables and a number. A **linear equation** in one (unknown) variable x is an equation of the form

Such equations are called 'linear' because they are closely related to straight lines.

$$ax + b = p,$$

where a, b, x and p are variables, with $a \neq 0$. For example, the equation

$$2x + 1 = 85$$

is linear in x. To solve this equation, we rearrange it as an equivalent equation such that x is the subject – isolated on the left – as follows. First, subtract 1 from both sides to obtain the equivalent equation

Strictly speaking, these are two equivalent equations; this is a common way of stringing equations together.

$$2x = 85 - 1 = 84,$$

then divide both sides by 2 to give the solution

$$x = \frac{84}{2} = 42.$$

For the general linear equation $ax + b = p$, we can express the unknown variable x in terms of the variables a, b and p in a similar manner; the final result is $x = (p - b)/a$.

A more challenging example is to solve the equation

$$\frac{3}{x+1} = \frac{5}{x+3}. \tag{4.1}$$

This equation makes sense only if $x + 1$ and $x + 3$ are non-zero; that is, if $x \neq -1$ and $x \neq -3$. Assuming these conditions, we can multiply both sides of equation (4.1) by the product $(x + 1)(x + 3)$ to obtain

The symbol $\neq$ means 'is not equal to'.

$$\frac{3(x+1)(x+3)}{x+1} = \frac{5(x+1)(x+3)}{x+3}.$$

By cancelling common factors on both sides, we obtain

$$3(x + 3) = 5(x + 1).$$

This equation is essentially linear, and can be solved by the following simple rearrangements:

$3x + 9 = 5x + 5$ (multiply out brackets on each side);

$4 = 2x$ (subtract 5 and $3x$ from each side);

$x = 2$ (divide each side by 2, and exchange the two sides).

Thus the solution is $x = 2$. Notice that for this value of x, both $x + 1$ and $x + 3$ are indeed non-zero and, moreover, both sides of equation (4.1) are equal (to 1), as expected.

It is good practice to check that your solution satisfies the original equation.

Remark You may be used to a less formal approach to several of these rearrangements, which can be described as

taking an expression to the other side and reversing its role.

For example, you have just seen the following.

◇ The equation $\dfrac{3}{x+1} = \dfrac{5}{x+3}$ became $3(x+3) = 5(x+1)$.

◇ $+\,5$ on the right became $-\,5$ on the left, and $+\,3x$ on the left became $-\,3x$ on the right.

◇ $\times\,2$ on the right became $\div\,2$ on the left.

The first of these informal procedures is often called 'cross-multiplying'. It can be stated in general as

$$\frac{a}{b} = \frac{c}{d} \quad \text{is equivalent to} \quad ad = bc,$$

provided that $b \neq 0$ and $d \neq 0$.

If you are already confident about this informal approach, then continue to use it. However, if you are at all uncertain about a rearrangement, then it is safest to 'do the same to both sides'.

Simultaneous linear equations

If more than one equation has to be satisfied by the unknowns in a problem, then these equations are called **simultaneous equations**. For example,

$$\begin{aligned} 2x + 3y &= 5, \\ 5x - 2y &= -16, \end{aligned} \tag{4.2}$$

are two simultaneous linear equations involving the unknowns x and y. The solutions for x and y must satisfy both equations at once.

There are several algebraic methods of solving such simultaneous linear equations, two of which are now reviewed briefly.

These two methods are very similar, but sometimes one is more convenient than the other.

Substitution

In this first method, we do the following.

◇ Rearrange one of the equations so that one unknown is equal to an expression involving the other.

◇ Substitute this expression in the other equation.

◇ Solve the resulting linear equation for the other unknown.

◇ Substitute this solution into either of the original equations to find the remaining unknown.

Either equation and either unknown may be chosen.

For example, with equations (4.2) we can work as follows.

◇ Rearrange the first equation, $2x + 3y = 5$, to give

$$y = \tfrac{1}{3}(5 - 2x).$$

◇ Substitute for y in the second equation, to give

$$5x - \tfrac{2}{3}(5 - 2x) = -16; \quad \text{that is,} \quad (5 + \tfrac{4}{3})x - \tfrac{10}{3} = -16.$$

◇ Solve the above equation, to give $x = -2$.

◇ Substitute $x = -2$ in the first equation, $2x + 3y = 5$, to find $y = 3$.

Thus the solutions are $x = -2$ and $y = 3$.

Check:
$2x + 3y$
$= 2 \times (-2) + 3 \times 3 = 5,$
$5x - 2y$
$= 5 \times (-2) - 2 \times 3 = -16.$

Elimination

In this second method, we do the following.

Either unknown may be chosen.

- ◇ Multiply the two equations by numbers chosen so that one of the unknowns has the same coefficient, possibly with the opposite sign, in both equations.
- ◇ Subtract or add the new equations to eliminate that unknown.
- ◇ Solve the resulting linear equation for the other unknown.
- ◇ Substitute this solution into either of the original equations to find the remaining unknown.

For example, with equations (4.2) we can work as follows.

- ◇ Multiply the first equation by 5 and the second by 2, so that x has the same coefficient in each new equation:

  $$10x + 15y = 25,$$
  $$10x - 4y = -32.$$

- ◇ Subtract the second of *these* equations from the first, to give

  $$15y - (-4y) = 25 - (-32); \quad \text{that is,} \quad 19y = 57.$$

- ◇ Solve the above equation, to give $y = 3$.
- ◇ Substitute $y = 3$ in $2x + 3y = 5$, to find $x = -2$.

For these simultaneous equations, elimination is more convenient.

Thus, as before, the solutions are $x = -2$ and $y = 3$.

Quadratic equations

A **quadratic expression** in the variable x is an algebraic expression which may be put in the form

$$ax^2 + bx + c,$$

The case $a = 0$ is excluded because the expression would then be linear (unless $b = 0$ also).

where $a \neq 0$. Correspondingly, a **quadratic equation** in the unknown x is an equation of the form

$$ax^2 + bx + c = p.$$

It is always possible, however, to subtract p from both sides of this equation and hence produce an equivalent quadratic equation which has 0 on the right-hand side. We shall therefore assume that this has been done from the outset, and confine attention to quadratic equations of the form

$$ax^2 + bx + c = 0, \quad \text{where } a \neq 0.$$

For example, one such equation is

$$x^2 + 5x - 14 = 0.$$

Here $a = 1$, $b = 5$ and $c = -14$.

Solving quadratic equations is more complicated than solving linear equations. There are various methods, one of which is *factorisation.*

Factorising a quadratic Some quadratic expressions with integer coefficients can be conveniently expressed as a product of two simple linear factors. For example,

$$x^2 + 5x + 6 = (x + 2)(x + 3),$$
$$5x^2 - 3x - 2 = (5x + 2)(x - 1),$$
$$x^2 + 4x + 4 = (x + 2)^2.$$

When a quadratic equation is expressed in factorised form, it is straightforward to solve the equation. For example, the two sides of the equation

$$(5x+2)(x-1)=0$$

are equal when either $5x+2=0$ or $x-1=0$. Therefore this equation has two solutions: $x=-2/5$ and $x=1$.

It is good practice to check solutions, but substituting solutions in the original equation(s) can be a time-consuming task!

Factorisation is a very effective way to solve a quadratic equation, *when it can be done.* But deciding whether a convenient factorisation is available can be tricky, and requires a level of skill which develops only with practice. Consider the special case where $a=1$, and b and c are integers. We then seek linear factors of x^2+bx+c of the form $x+\alpha$ and $x+\beta$. Now

α and β are the Greek letters alpha and beta. All the Greek letters are listed in *Guide to Preparation.*

$$(x+\alpha)(x+\beta)=x^2+(\alpha+\beta)x+\alpha\beta.$$

Comparing this right-hand side with x^2+bx+c, we see that α and β must be chosen so that

$$\alpha+\beta=b \quad\text{and}\quad \alpha\beta=c.$$

Therefore, for a simple factorisation, we need to consider the pairs of integers α and β whose product is c, and check whether any of these have sum b. For example, to solve

$$x^2+5x-14=0$$

by factorisation, we need to find two integers whose product is $c=-14$ and whose sum is $b=5$. After considering the various possibilities, the values 7 and -2 are found to satisfy both of these conditions, so we obtain the factorisation

The possible pairs are:

$14, -1$; $7, -2$;
$1, -14$; $2, -7$.

Of these, only the pair 7, -2 has sum 5:

$$7+(-2)=5.$$

$$x^2+5x-14=(x+7)(x-2).$$

Thus the solutions of the equation $x^2+5x-14=0$ are $x=-7$ and $x=2$.

Since such a convenient factorisation is not always available, we need another approach.

Solving quadratic equations by the formula Fortunately, there is a formula which enables us to find the solutions of *any* quadratic equation. This states that the solutions of the equation

$$ax^2+bx+c=0 \quad (a\neq 0) \tag{4.3}$$

are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}. \tag{4.4}$$

This formula is derived later in the course.

Here the symbol $\pm$ denotes two alternative solutions, one with the plus sign and the other with the minus sign.

For example, to solve the equation

$$x^2-x-1=0,$$

we use equation (4.4) with $a=1$, $b=-1$ and $c=-1$, to obtain

$$x=\frac{-(-1)\pm\sqrt{(-1)^2-4\times 1\times(-1)}}{2\times 1}=\tfrac{1}{2}(1\pm\sqrt{5}).$$

The decimal values of these two solutions are

$$\tfrac{1}{2}(1+\sqrt{5})=1.618 \text{ (to 3 d.p.)}, \quad \tfrac{1}{2}(1-\sqrt{5})=-0.618 \text{ (to 3 d.p.)}.$$

The fact that these solutions are not integers confirms that there was no simple factorisation of the quadratic expression x^2-x-1.

A 'real solution' is a solution which is a real number.

Equation (4.4) indicates that some quadratic equations have real solutions, but others do not. For real solutions, the quantity beneath the square root sign must be positive or 0. Hence there are:

two real solutions if $b^2 - 4ac > 0$;
one real solution if $b^2 - 4ac = 0$;
no real solutions if $b^2 - 4ac < 0$.

For example, the equation $x^2 + 4x + 4 = 0$ has one real solution ($x = -2$), because in this case

$$b^2 - 4ac = 4^2 - 4 \times 1 \times 4 = 0.$$

On the other hand, the equation $x^2 + 2x + 3 = 0$ has no real solutions, because in this case

$$b^2 - 4ac = 2^2 - 4 \times 1 \times 3 = -8.$$

Remark Solutions of an equation are often called *roots* of the equation. In the case of a quadratic equation with $b^2 - 4ac = 0$, the one solution is called a *repeated solution* or *repeated root.*

The next activity asks you to practise solving linear and quadratic equations.

Activity 4.3 Solving equations

(a) Solve the equation

$$2x - 3 = 7.$$

(b) Solve the following equation to find x in terms of a and b, assuming that $a \neq b$:

$$a(x + a) = b(x + b).$$

(c) Solve the pair of simultaneous equations

$$2x + 6y = -11,$$
$$4x - 3y = -2.$$

(d) Solve each of the following quadratic equations.

(i) $x^2 + 2x - 3 = 0$

(ii) $2x^2 + 6x - 5 = 0$

(iii) $6x^2 - 13x + 6 = 0$

Solutions are given on page 44.

Later in the course you will see how Mathcad is used to solve equations. In particular, you will learn how to use the **Symbolic** menu item **Solve for Variable** to solve quadratic equations.

Summary of Section 4

This section recalled basic rearrangements of algebraic expressions and equations, and showed you how some of these rearrangements can be carried out with Mathcad. In particular, you saw:

- ◇ the notation for algebraic expressions and equations;
- ◇ the rules for rearranging algebraic expressions in order to arrive at equivalent expressions;
- ◇ the rules for rearranging equations in order to arrive at equivalent equations;
- ◇ how to solve simultaneous linear equations by the methods of substitution and elimination;
- ◇ how to solve a quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, by the method of factorisation and by using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

- ◇ the use of **Simplify**, **Expand Expression** and **Factor Expression** on Mathcad's **Symbolic** menu.

Exercises for Section 4

Exercise 4.1

Use hand calculation, your calculator or Mathcad, as appropriate, to do each of the following.

To use Mathcad, you will need to create a new, blank document.

(a) Collect like terms in the expression

$$3x - 2y + 4z + 6x + 2y - 3z - 2x - 3y + z.$$

(b) Cancel common factors in the fraction

$$\frac{8a^2bc^2}{6a^2c}.$$

(c) Multiply out each of the following products.

(i) $(x - 1)(y + 2)$

(ii) $(x - 1)^2(y + 2)^2$

(d) Factorise the following expressions.

(i) $4u^2 - 9$

(ii) $5ax^2 - 10bx$

(iii) $x^2 - 4x + 4$

(iv) $a^3 - b^3$

(e) Combine the following fractions:

$$\frac{1}{x+2} - \frac{2}{2x+5}.$$

(f) Expand the following fraction, simplifying each fraction of the resulting sum:

$$\frac{a^4+1}{a^2}.$$

Exercise 4.2

Use hand calculation or your calculator, as appropriate, to do each of the following.

(a) Solve the following equation to find x in terms of a and b, assuming that $a \neq b$:

$$(x+a)b = ax.$$

(b) Solve the pair of simultaneous equations

$$3a + 5b = 21,$$
$$2a + 3b = 13.$$

(c) Solve each of the following equations.

(i) $x^2 + x - 12 = 0$

(ii) $1.1x^2 + 0.99x - 12.01 = 0$

Summary of Chapter A0

After studying a chapter, it is suggested that you spend some time reviewing the material in order to consolidate your knowledge. To help you in this, each chapter's key points are listed in a summary like the one below. There are many different ways to review material. Some people like to list the words and notation that they can remember and indicate connections between those that are related. Others like to rehearse the 'story' of the material, perhaps saying it out loud (possibly recording it on tape) or telling it to a friend.

Whatever method you use, it is suggested that you make brief notes about how you have coped with different aspects of the material (reading the text, doing algebra, working with the computer, and so on). In particular, you may wish to write down those aspects on which you need to seek advice from your tutor or to discuss with other students.

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Natural numbers, integers, real numbers, rational and irrational numbers, finite, infinite and recurring decimals, real line, powers, variable, identity, algebraic expression, term, coefficient, linear expression, linear equation, quadratic expression, quadratic equation, factorisation.

Symbols and notation to know and use

$\mathbb{N}$ (natural numbers), $\mathbb{Z}$ (integers), $\mathbb{Q}$ (rational numbers),
$\mathbb{R}$ (real numbers);
dot notation for recurring decimals;
notation for powers, including roots.

Mathematical skills

- ◇ Evaluate arithmetic expressions, by hand or calculator.
- ◇ Use variables.
- ◇ Rearrange algebraic expressions.
- ◇ Solve simultaneous linear equations and quadratic equations.
- ◇ Determine the number of real solutions of a quadratic equation.

Mathcad skills

- ◇ Access Mathcad and the course files (having loaded the software onto your hard disk).
- ◇ Create, navigate around, save, print and close a Mathcad document.
- ◇ Use Mathcad for basic calculations and basic algebra.

Ideas to be aware of

- ◇ Whether to use hand calculation, your calculator or Mathcad may depend on the type of calculation to be performed.
- ◇ The equals sign = has several different meanings.

Appendix: Numerical form of answers

In Section 3, you were reminded that there are several ways of representing real numbers, including ordinary decimals, scientific notation, fractions and roots, and that decimal representations may be rounded either to a given number of decimal places or to a given number of significant figures. One consequence of this is that you will have to decide how to present numerical answers to questions. The following guidelines are offered to help you with this.

The course will attempt to follow these guidelines, but may not always succeed!

- ◇ Always check to see whether a question (especially one on an assignment or examination) asks for numerical answers in a particular form.
- ◇ If no particular form of answer is specified, then decimal answers will usually be rounded to three decimal places or three significant figures.
- ◇ In questions where the calculations involve only simple fractions, then answers may be given in this form.
- ◇ Even if you are going to round an *answer*, try to perform all the calculations to the full accuracy of whatever method (hand, calculator, Mathcad) you are using. Round answers only at the final stage of presenting them; in particular, do *not* use rounded answers to earlier parts of a question in subsequent calculations (unless instructed to do so).

For example, in the solutions for this chapter, numbers obtained from a calculator are recorded to eight significant figures.

Solutions to Activities

In these solutions, all decimals obtained by a calculator are given to eight significant figures.

Solution 3.1

(a) Since $625 = 25^2 = 5^4$, we have

$$625^{3/4} = \left(\sqrt[4]{625}\right)^3 = 5^3 = 125.$$

(b) By calculator,

$$\frac{78}{98} = 0.795\,918\,37 = 0.795\,918 \text{ (to 6 d.p.)}.$$

(c) By calculator,

$$2^{4/3} = 2.519\,842\,1 = 2.520 \text{ (to 3 d.p.)}.$$

(d) By calculator,

$$\begin{aligned} 2^{100} &= 1.267\,650\,6 \times 10^{30} \\ &= 1.27 \times 10^{30} \text{ (to 3 s.f.)}. \end{aligned}$$

Solution 3.3

(a) For $r = 2$,

$$\begin{aligned} C &= 2\pi r \\ &= 2 \times \pi \times 2 \\ &= 12.566\,371 = 12.6 \text{ (to 3 s.f.)}. \end{aligned}$$

For $r = 5$,

$$\begin{aligned} C &= 2\pi r \\ &= 2 \times \pi \times 5 \\ &= 31.415\,927 = 31.4 \text{ (to 3 s.f.)}. \end{aligned}$$

(b) For $r = 1.5$,

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \times (1.5)^2 \\ &= 7.068\,583\,5 = 7.07 \text{ (to 3 s.f.)}. \end{aligned}$$

For $r = 3.7$,

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \times (3.7)^2 \\ &= 43.008\,403 = 43.0 \text{ (to 3 s.f.)}. \end{aligned}$$

(c) For $r = 1.5$ and $h = 8$,

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi \times (1.5)^2 \times 8 \\ &= 56.548\,668 = 56.5 \text{ (to 3 s.f.)}. \end{aligned}$$

(d) For $a = 1$ and $b = 2$,

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{1 + 4} \\ &= 2.236\,068\,0 = 2.24 \text{ (to 3 s.f.)}. \end{aligned}$$

For $a = 3$ and $b = 4$,

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{9 + 16} \\ &= 5. \end{aligned}$$

Solution 4.1

(a) $(a - 2b + 3c - 4d) - (-4a + 3b - 2c + d)$
$= 5a - 5b + 5c - 5d$

(b) $\dfrac{20a^2b}{15ab^2} = \dfrac{4a}{3b}$

(c) (i) $\begin{aligned}[t] (x + 1)(x - 2) &= x^2 - 2x + x - 2 \\ &= x^2 - x - 2 \end{aligned}$

(ii) $(a - b)^2 = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2$

(iii) Multiplying out the first two factors first gives

$$\begin{aligned} &(t + 1)(t + 2)(t + 3) \\ &= (t^2 + 3t + 2)(t + 3) \\ &= t^3 + 3t^2 + 3t^2 + 9t + 2t + 6 \\ &= t^3 + 6t^2 + 11t + 6. \end{aligned}$$

Alternatively, the last two factors could be multiplied first.

(d) (i) The term ab is a common factor, so

$$a^2b - ab^2 = ab(a - b).$$

(ii) The expression $x^2 - 4$ is a difference of two squares, so

$$x^2 - 4 = (x - 2)(x + 2).$$

(iii) The expression $x^2 - 2x + 1$ has the same form as $a^2 - 2ab + b^2$, obtained in part (c)(ii), with x instead of a and 1 instead of b, so

$$x^2 - 2x + 1 = (x - 1)^2.$$

(e) A common denominator is $3(x + 2)(x + 3)$, so

$$\begin{aligned} \frac{4}{3(x + 2)} + \frac{2}{x + 3} &= \frac{4(x + 3) + 2 \times 3(x + 2)}{3(x + 2)(x + 3)} \\ &= \frac{10x + 24}{3(x + 2)(x + 3)} \\ &= \frac{2(5x + 12)}{3(x + 2)(x + 3)}. \end{aligned}$$

(f) Dividing each term of the numerator by the denominator, and simplifying, gives

$$\frac{x^2 + 2x + 1}{x^3} = \frac{1}{x} + \frac{2}{x^2} + \frac{1}{x^3}.$$

Solution 4.3

(a) Starting from

$$2x - 3 = 7,$$

use the following rearrangements:

$$2x = 10 \quad \text{(add 3 to both sides);}$$
$$x = 5 \quad \text{(divide both sides by 2).}$$

(b) Starting from

$$a(x + a) = b(x + b),$$

use the following rearrangements:

$$ax + a^2 = bx + b^2 \quad \text{(multiply out brackets);}$$
$$(a - b)x = b^2 - a^2 \quad \text{(collect like terms);}$$
$$(a - b)x = (b - a)(b + a) \quad \text{(difference of two squares).}$$

The assumption $a \neq b$ means $a - b \neq 0$, so we can divide both sides by $a - b$, to give

$$\begin{aligned} x &= \frac{(b-a)(b+a)}{a-b} \\ &= -\frac{(a-b)(b+a)}{a-b} \\ &= -(b+a) = -a - b. \end{aligned}$$

(c) To solve

$$2x + 6y = -11, \tag{1}$$
$$4x - 3y = -2, \tag{2}$$

we first multiply equation (1) by 2 so that the coefficients of x are the same in both equations:

$$4x + 12y = -22, \tag{3}$$
$$4x - 3y = -2. \tag{4}$$

Then subtract equation (4) from equation (3), to give

$$15y = -20,$$

so $y = -4/3$.

On substituting this value for y into equation (1), we obtain

$$2x = -11 - 6 \times (-4/3) = -3,$$

so $x = -3/2$.

Thus the solutions are $x = -3/2$ and $y = -4/3$.

(d) (i) To solve

$$x^2 + 2x - 3 = 0,$$

either note the factorisation

$$x^2 + 2x - 3 = (x - 1)(x + 3),$$

so the solutions are $x = 1$ and $x = -3$, or use the formula, giving

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-3)}}{2 \times 1} \\ &= \tfrac{1}{2}(-2 \pm \sqrt{16}) \\ &= \tfrac{1}{2}(-2 \pm 4) \\ &= -1 \pm 2, \end{aligned}$$

so $x = 1$ and $x = -3$, once again.

(ii) To solve

$$2x^2 + 6x - 5 = 0$$

(assuming that no convenient factorisation occurs to you), use the formula:

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4 \times 2 \times (-5)}}{2 \times 2} \\ &= \tfrac{1}{4}(-6 \pm \sqrt{76}) \\ &= \tfrac{1}{4}(-6 \pm \sqrt{4 \times 19}) \\ &= \tfrac{1}{4}(-6 \pm 2\sqrt{19}) \\ &= \tfrac{1}{2}(-3 \pm \sqrt{19}). \end{aligned}$$

In decimals, the solutions are $x = 0.679$ (to 3 d.p.) and $x = -3.679$ (to 3 d.p.).

(In the above calculation, the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

has been used, with $ab = 76$, $a = 4$ and $b = 19$.)

(iii) To solve

$$6x^2 - 13x + 6 = 0$$

(assuming that no convenient factorisation occurs to you), use the formula:

$$\begin{aligned} x &= \frac{-(-13) \pm \sqrt{(-13)^2 - 4 \times 6 \times 6}}{2 \times 6} \\ &= \tfrac{1}{12}(13 \pm \sqrt{25}) \\ &= \tfrac{1}{12}(13 \pm 5), \end{aligned}$$

giving $x = 18/12 = 3/2$ and $x = 8/12 = 2/3$.

(The fact that these solutions are fractions indicates that there is a simple factorisation, namely

$$6x^2 - 13x + 6 = (3x - 2)(2x - 3),$$

but it might have taken some time to find this!)

Solutions to Exercises

Solution 3.1

(a) (i) Since $49 = 7^2$, we have

$$49^{3/2} = \left(\sqrt{49}\right)^3 = 7^3 = 343.$$

(ii) By calculator,

$$50^{3/2} = 353.553\,39 = 354 \text{ (to 3 s.f.)}.$$

(iii) Since $26 = 2 \times 13$ and $65 = 5 \times 13$, we can cancel the factor 13 in the numerator and denominator, to give

$$\frac{26}{65} = \frac{2 \times 13}{5 \times 13} = \frac{2}{5} = 0.4.$$

(iv) By calculator,

$$\frac{27}{65} = 0.415\,384\,62 = 0.415 \text{ (to 3 s.f.)}.$$

(v) By calculator,

$$\begin{aligned} 3^{-10} &= 0.000\,016\,935\,088 \\ &= 1.69 \times 10^{-5} \text{ (to 3 s.f.)}. \end{aligned}$$

Solution 3.2

(a) (i) For $r = 2.5$,

$$\begin{aligned} V &= \tfrac{4}{3}\pi r^3 \\ &= \tfrac{4}{3} \times \pi \times (2.5)^3 \\ &= 65.449\,847 = 65.4 \text{ (to 3 s.f.)}. \end{aligned}$$

For $r = 4.7$,

$$\begin{aligned} V &= \tfrac{4}{3}\pi r^3 \\ &= \tfrac{4}{3} \times \pi \times (4.7)^3 \\ &= 434.892\,77 = 435 \text{ (to 3 s.f.)}. \end{aligned}$$

(ii) For $a = 2$, $b = 3$ and $c = 4$,

$$s = \tfrac{1}{2}(a + b + c) = \tfrac{1}{2}(2 + 3 + 4) = 4.5$$

and

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{4.5 \times 2.5 \times 1.5 \times 0.5} \\ &= \sqrt{8.4375} \\ &= 2.904\,737\,5 = 2.90 \text{ (to 3 s.f.)}. \end{aligned}$$

Solution 4.1

(a) The expression

$$3x - 2y + 4z + 6x + 2y - 3z - 2x - 3y + z$$

has terms in x, y and z, which can be collected to give

$$7x - 3y + 2z.$$

(b) The numerator and denominator have common factors 2, a^2 and c, so

$$\frac{8a^2bc^2}{6a^2c} = \frac{4bc}{3} = \tfrac{4}{3}bc.$$

(c) (i) We have

$$(x-1)(y+2) = xy + 2x - y - 2.$$

(ii) To expand

$$(x-1)^2(y+2)^2 = (x^2 - 2x + 1)(y^2 + 4y + 4)$$

involves a more complicated calculation, and there is a good chance of making a mistake. This seems a good time to use Mathcad's **Expand Expression**, which gives the answer

$$\begin{aligned} &x^2y^2 + 4x^2y + 4x^2 - 2xy^2 \\ &- 8xy - 8x + y^2 + 4y + 4. \end{aligned}$$

(d) (i) The expression $4u^2 - 9$ is a difference of two squares:

$$4u^2 - 9 = (2u)^2 - 3^2.$$

Therefore

$$4u^2 - 9 = (2u - 3)(2u + 3).$$

(ii) The terms of $5ax^2 - 10bx$ have common factors 5 and x, so

$$5ax^2 - 10bx = 5x(ax - 2b).$$

(iii) The expression $x^2 - 4x + 4$ is of the form $a^2 - 2ab + b^2$, with x instead of a and 2 instead of b. Thus

$$x^2 - 4x + 4 = (x - 2)^2.$$

(iv) The expression $a^3 - b^3$ seems difficult to factorise (unless you have seen it factorised before). Mathcad's **Factor Expression** gives

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2),$$

which you can check by multiplying out the brackets on the right.

(e) A common denominator is $(x+2)(2x+5)$, so

$$\begin{aligned} \frac{1}{x+2} - \frac{2}{2x+5} &= \frac{(2x+5) - 2(x+2)}{(x+2)(2x+5)} \\ &= \frac{1}{(x+2)(2x+5)}. \end{aligned}$$

(f) We have

$$\frac{a^4 + 1}{a^2} = \frac{a^4}{a^2} + \frac{1}{a^2} = a^2 + \frac{1}{a^2}.$$

Solution 4.2

(a) Starting from

$$(x+a)b = ax,$$

use the following rearrangements:

$$bx + ab = ax \quad \text{(multiply out brackets);}$$
$$(b-a)x = -ab \quad \text{(collect like terms).}$$

Since $a \neq b$, we can divide both sides by $b-a$, to give

$$x = \frac{-ab}{b-a} = \frac{ab}{a-b}.$$

(b) To solve

$$3a + 5b = 21, \qquad (1)$$
$$2a + 3b = 13, \qquad (2)$$

we multiply equation (1) by 2 and equation (2) by 3, so that the coefficients of a are the same:

$$6a + 10b = 42, \qquad (3)$$
$$6a + 9b = 39. \qquad (4)$$

Then subtract equation (4) from equation (3), to give $b = 3$.

On substituting this value for b into equation (1), we obtain

$$3a = 21 - 5 \times 3 = 6,$$

so $a = 2$.

Thus the solutions are $a = 2$ and $b = 3$.

(c) (i) To solve

$$x^2 + x - 12 = 0,$$

either note the factorisation

$$x^2 + x - 12 = (x+4)(x-3),$$

so the solutions are $x = -4$ and $x = 3$, or use the formula, giving

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times (-12)}}{2 \times 1}$$
$$= \tfrac{1}{2}(-1 \pm \sqrt{49})$$
$$= \tfrac{1}{2}(-1 \pm 7),$$

so $x = 3$ and $x = -4$, once again.

(ii) To solve

$$1.1x^2 + 0.99x - 12.01 = 0,$$

we use the formula (a convenient factorisation seems most unlikely!):

$$x = \frac{-0.99 \pm \sqrt{0.99^2 - 4 \times 1.1 \times (-12.01)}}{2 \times 1.1}$$
$$= \frac{-0.99 \pm \sqrt{0.9801 + 52.844}}{2.2}$$
$$= \frac{-0.99 \pm \sqrt{53.8241}}{2.2}$$
$$= \frac{-0.99 \pm 7.336\,491\,0}{2.2},$$

giving

$$x = \frac{6.346\,491\,0}{2.2} = 2.884\,768\,6$$
$$= 2.88 \text{ (to 3 s.f.)}$$

and

$$x = \frac{-8.326\,491\,0}{2.2} = -3.784\,768\,6$$
$$= -3.78 \text{ (to 3 s.f.).}$$

(Notice that the coefficients in the two quadratic equations in parts (i) and (ii) are quite close, and the solutions are similarly close; this provides one possible check on the working.)

Index

The Open University
ISBN 0 7492 9314 4